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Marriage in Denumerable Societies

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A *society* is an ordered triple (M, W, K) of sets such that M, W are disjoint and $K \subseteq M \times W$. An *espousal* of (M, W, K) is a subset of K of the form $\{(a, e(a)) : a \in M\}$ where $e(a_1) \neq e(a_2)$ whenever $a_1 \neq a_2$. If M is countable, we associate with (M, W, K) and each ordinal α a function m_α from the set of subsets of W into the union of the set of integers and $\{-\infty, \infty\}$. Three different definitions of m_α (all fairly elaborate) are presented and their equivalence under suitable conditions is proved. Assuming M to be countable, we prove that (i) (M, W, K) has an espousal if and only if $m_\Omega(X) \geqslant 0$ for every subset X of W, where Ω is the first uncountable ordinal, and (ii) if $X \subseteq W$ and $\alpha \leqslant \beta$ and $m_\alpha(X) < \infty$ and $m_\alpha(Z) \geqslant 0$ for every subset Z of X then $m_\alpha(Z) = m_\beta(Z)$ for every subset Z of X. The result (i) is a theorem of Damerell and Milner, but the proof here presented differs somewhat in formulation and structure from theirs.

1. Statement and Background of the Problem

Some set-theoretic conventions, although many of them are standard, will first be stated for clarity. A *relation* is a set of ordered pairs. Let Q, R be relations, A, B be sets and a be an element. Then $|A|$ denotes the cardinal number of A, 2^A denotes the set of all subsets of A, R^{-1} denotes $\{(x, y): (y, x) \in R\}$, $R\langle a\rangle$ denotes $\{y: (a, y) \in R\}$, $R(a)$ denotes the element of $R\langle a\rangle$ when $|R\langle a\rangle| = 1$, $R[A]$ denotes $\bigcup_{x\in A} R\langle x\rangle$ and $R \circ Q$ denotes $\{(x, y): y \in R[Q\langle x\rangle]\}$. The *domain* dom R and *range* rge R of R are $\{x: (x, t) \in R$ for some $t\}$ and $\{y: (t, y) \in R$ for some $t\}$, respectively. If $C \subseteq \operatorname{dom} R$, then $R \mid C$ denotes the restriction $R \cap (C \times \operatorname{rge} R)$ of R to C. We call R a *function* if $|R\langle x\rangle| = 1$ for every $x \in \operatorname{dom} R$. A *function f from A into B*, or function $f: A \to B$, is a function f such that dom $f = A$, rge $f \subseteq B$. A *function from A onto B* is a function with domain A and range B. A function f is *one-to-one* if f^{-1} is a function.

Borrowing a term from [12], we define a *society* to be an ordered triple (M, W, K) such that M, W are disjoint sets and K is a subset of $M \times W$. An *espousal* of a society (M, W, K) is a one-to-one function $e: M \to W$ such that $e \subseteq K$. This paper is a contribution to the study of the problem, "Which societies have espousals?" If we think of elements of M as men, elements of W as women and K as the set of ordered pairs (m, w) such that m knows w, then we are asking in which societies wives can be found for all the men so that each man is married to a woman he knows.

Some equivalent and related problems. We could drop the requirement that M and W be disjoint by defining a *community* to be an ordered triple (M, W, K) of sets such that $K \subseteq M \times W$, and an *espousal* of this community to be a one-to-one function $e: M \to W$ such that $e \subseteq K$. The question of which societies have espousals is equivalent to the apparently more general question of which communities have espousals because, if (M, W, K) is a community and f is any one-to-one function from W onto a set disjoint from M, then the existence of an espousal e of the society $(M, f[W], f \circ K)$ implies the existence of an espousal $f^{-1} \circ e$ of the community (M, W, K) and, conversely, the existence of an espousal e' of (M, W, K) implies the existence of an espousal $f \circ e'$ of $(M, f[W], f \circ K)$.

A *transversal* or *system of distinct representatives* of a family of sets $(A_i: i \in I)$ is a family $(a_i: i \in I)$ of elements such that $a_i \in A_i$ for every $i \in I$ and a_i, a_j are distinct whenever $i \neq j$. The existence of a transversal $(a_i: i \in I)$ of $(A_i: i \in I)$ is equivalent to the existence of an espousal $\{(i, a_i): i \in I\}$ of the community

$$\Big(I, \bigcup_{i \in I} A_i, \bigcup_{i \in I} (\{i\} \times A_i)\Big);$$

and a community (M, W, K) has an espousal iff the family of sets $(K\langle x \rangle: x \in M)$ has a transversal. Hence enquiring which families of sets have transversals is equivalent to enquiring which communities (or, equivalently, which societies) have espousals.

A *matching* in a graph G is a subset L of the set $E(G)$ of edges of G such that no vertex of G is incident with more than one element of L. If X is a subset of the set $V(G)$ of vertices of G, we define an *X-matching* in G to be a matching in G of the form $\{\lambda(x): x \in X\}$, where each $x \in X$ is joined by $\lambda(x)$ to some element of $V(G) \backslash X$. From any society (M, W, K) we can construct a bipartite graph B by letting $V(B) = M \cup W$ and $E(B) = \{\mu(u, v): (u, v) \in K\}$, where, for each $(u, v) \in K$, $\mu(u, v)$ is an edge joining u to v; and conversely any bipartite graph without multiple edges is constructible in this way from a suitable society. The fact that there exists an M-matching in B iff (M, W, K) has an espousal provides an obvious and well known translation of our original problem into one about existence of matchings in bipartite graphs.

Let D be a digraph. Let $K(D)$ be the set of all ordered pairs $(x, y) \in V(D) \times V(D)$ such that at least one edge of D has tail x and head y. Then the community $(V(D), V(D), K(D))$ has an espousal iff $E(D)$ has a subset L such that each vertex of D is the tail of exactly one and the head of at most one element of L, or, equivalently, iff D has a spanning subdigraph (i.e., a subdigraph including all the vertices of D) each of whose connected components is either a directed path which has no terminal vertex (but may have an initial vertex) or a directed circuit. Thus the problem of which digraphs have spanning subdigraphs of this kind is contained in the problem of which societies have espousals.

Define a *matrimony* of a community (M, W, K) to be a one-to-one function e from M *onto* W such that $e \subseteq K$. In particular, a matrimony of a society may be thought of as a prescription for finding each man a wife whom he knows in such a way that all the women as well as all the men get married. A *perfect matching* in a graph G is a subset L of $E(G)$ such that each vertex of G is incident with exactly one element of L. If B is the bipartite graph constructed from a society (M, W, K) as described above, then clearly (M, W, K) has a matrimony iff B has a perfect matching. But it is fairly easy to prove that *B has a perfect matching iff there exist both an M-matching and a W-matching in B.* For any perfect matching in B is both an M-matching and a W-matching; and conversely, if there exist an M-matching L_1 and a W-matching L_2 in B, then the subgraph H of B such that $V(H) = M \cup W$, $E(H) = L_1 \cup L_2$ has a perfect matching (L, say) since each connected component of H is either a graph with just one edge or a circuit of even length or an infinite path, and L is a perfect matching in B also. Since there exists an M-matching in B iff the society (M, W, K) has an espousal and there exists a W-matching in B iff the society (W, M, K^{-1}) has an espousal, we conclude that (M, W, K) has a matrimony iff both (M, W, K) and (W, M, K^{-1}) have espousals. Thus determination of which societies have espousals would also determine which societies have matrimonies. (In effect, this paragraph recalls some observations from [11, Chap. VI, Sect. 3] and [4].)

The problem of which societies have matrimonies is essentially equivalent to the apparently more general problem of which communities have matrimonies: this statement is justified in the same way as the corresponding statement about espousals. The problem [19] of which digraphs have a spanning subdigraph in which each vertex has invalency 1 and outvalency 1 (i.e., a spanning subdigraph whose connected components are directed paths extending to infinity in both directions and/or directed circuits) is tantamount to asking for which digraphs D the community $(V(D), V(D), K(D))$ has a matrimony.

History. König [9, 10, 11] showed that a bipartite graph B contains a finite matching consisting of n edges iff $V(B)$ has no subset U such that

$|U| < n$ and every edge of B is incident with an element of U. If (M, W, K) is a society and the set M is finite, König's theorem with $n = |M|$, applied to the bipartite graph constructed from (M, W, K), implies that (M, W, K) has an espousal iff

$$|K[A]| \geqslant |A| \qquad \text{for every subset } A \text{ of } M, \tag{1}$$

a theorem established independently, in the language of transversals of families of sets, by P. Hall [7]. Subsequent authors [4, 6, 8] showed that (1) is necessary and sufficient for the existence of an espousal in any society (M, W, K) such that $K\langle a\rangle$ is finite for each $a \in M$. This result and our remark that (M, W, K) has a matrimony iff (M, W, K) and (W, M, K^{-1}) have espousals imply that a society (M, W, K) in which the sets $K\langle a\rangle$ $(a \in M)$ and $K^{-1}\langle x\rangle$ $(x \in W)$ are all finite has a matrimony iff $|K[A]| \geqslant |A|$ for every subset A of M and $|K^{-1}[S]| \geqslant |S|$ for every subset S of W, which is a theorem independently established by Rado [17] and is also equivalent to a theorem of de Bruijn [3] about common transversals of two partitions of the same set. If some sets $K\langle a\rangle$ may be infinite, the necessary condition (1) for the existence of an espousal is no longer sufficient; but a more complicated necessary and sufficient condition was obtained by Rado [18] for the case in which $K\langle a\rangle$ is infinite for just one $a \in M$ and by several authors [1, 5, 13, 21] for the case in which $K\langle a\rangle$ is infinite for finitely many elements a of M. In [15], I conjectured that a certain even more complicated condition would be necessary and sufficient for the existence of an espousal in any society in which the set $M \cup W$ is denumerable, and Damerell and Milner [2] proved this conjecture. In fact, they only needed to assume denumerability of M.

The present paper aims to develop certain aspects of the theory underlying this theorem. *Inter alia*, it will give its own self-contained proof of the theorem, which makes no claim to be superior to (or even necessarily as good as) that of [2] but, being a little different, may possibly yield its own insights. The conjecture in [15] was prompted by a description of the ideas of [18] given to me verbally by Professor Rado, and much of the subsequent work leading to the present paper was based on key ideas from [2], of which two preliminary drafts were made available to me by Dr. Damerell. I am much indebted to Professor Rado and the authors of [2] for stimulating this work.

In [16], another necessary and sufficient condition is given for the existence of an espousal in a society (M, W, K) such that M is denumerable. This condition, however, differs somewhat in character from the one discussed in [2], [15] and the present paper, since it presupposes some knowledge about espousals in special subsocieties of (M, W, K).

Further background can be found in [14].

Statement of the main theorem. Throughout this paper, the Axiom of Choice will be assumed. A set A will be called *denumerable* if $|A| = \aleph_0$, and *countable* if $|A| \leqslant \aleph_0$. The first infinite ordinal will be denoted by ω, and the first uncountable ordinal by Ω. The *successor* of an ordinal α is $\alpha + 1$. An ordinal will be called a *successor ordinal* if it is the successor of some ordinal, and a *limit ordinal* if it is neither 0 nor a successor ordinal.

Let $\mathscr{Q}$ denote a set whose elements are the integers and two further "numbers" ∞ and $-\infty$. Elements of $\mathscr{Q}$ will be called *quasi-integers.* The *size* $\| A \|$ of a set A is defined to be its cardinality $|A|$ if A is finite and to be ∞ if A is infinite: thus $\| A \| \in \mathscr{Q}$ for every set A. The *sum* $a_1 + \cdots + a_n$ of n quasi-integers $a_1, \ldots, a_n$ has its usual meaning if the a_i are all integers, is defined to be ∞ if at least one a_i is ∞, and is defined to be $-\infty$ if no a_i is ∞ but at least one is $-\infty$. The *difference* $a - b$ of two quasi-integers is the sum of a and $-b$; and likewise the sum of the quasi-integers a, $-b$, c may be denoted by $a - b + c$, etc. For our purposes, the most important distinctive feature of these definitions is that $\infty - \infty$ is defined to be ∞, since we wish to think of $\infty - \infty$ as the *largest possible* value of $\| A \backslash B \|$ for sets A, B such that $B \subseteq A$ and $\| A \| = \| B \| = \infty$. *Inequalities* between quasi-integers are defined in the obvious way. The *infimum* $\inf \mathscr{A}$ of a non-empty subset $\mathscr{A}$ of $\mathscr{Q}$ is the greatest quasi-integer q such that $q \leqslant a$ for every $a \in \mathscr{A}$, and the *supremum* $\sup \mathscr{A}$ is analogously defined.

An infinite sequence $T_1, T_2, \ldots$ of sets such that $T_1 \subseteq T_2 \subseteq \cdots$ will be called a *tower*, and will more specifically be said to be a tower *on* the set $T_1 \cup T_2 \cup \cdots$. A tower will be denoted by a bold face letter and its jth term by the corresponding italic letter with subscript j, e.g., $\mathbf{T}$ denotes the tower $T_1, T_2, \ldots$.

Let $\Gamma = (M, W, K)$ be a society. Elements of M will be called *men* of Γ and elements of W will be called *women* of Γ. If M is countable, Γ will be called *male-countable.* Throughout Sections 2–8 and the remainder of Section 1, it will be understood that we are discussing a male-countable society $\Gamma = (M, W, K)$ and the symbols Γ, M, W, K should be interpreted accordingly. Furthermore, α, β will always denote ordinal numbers, q will always denote a quasi-integer, X, Y will always denote subsets of W and g will always denote a function from 2^W into $\mathscr{Q}$. It will be understood that any theorem, lemma or corollary whose statement contains one or more of the symbols α, β, X, Y, g, q is asserted to be true for all choices of the entities represented by those symbols compatible with the foregoing notational convention: this will reduce the need for repetitious use of phraseology like "for every subset X of W and every function $g: 2^W \to \mathscr{Q}, \ldots$."

A tower $\mathbf{T}$ will be called *g-constant* if $g(T_1) = g(T_2) = \cdots$, and $\mathfrak{T}(X, g)$

will denote the set of all g-constant towers on X. If $\mathbf{T}$ is a g-constant tower, $\hat{g}(\mathbf{T})$ will denote the value of $g(T_j)$ for each j. We define $\tilde{g}(X)$ to be $\inf\{\hat{g}(\mathbf{T}): \mathbf{T} \in \mathfrak{T}(X, g)\}$. To see that this definition makes sense, we must observe that $\mathfrak{T}(X, g)$ is nonempty since it includes the tower $X, X, X, \ldots$.

We shall say that g *subdues* X *at* [*below*] q if every finite subset of X is contained in a subset Z of X such that $g(Z) = q$ [$g(Z) \leqslant q$]. We might have chosen to define $\tilde{g}(X)$ to be the infimum of the set of those quasi-integers at which g subdues X, or to be the infimum of the set of those quasi-integers below which g subdues X. These definitions are not in general equivalent to each other or to the definition of $\tilde{g}(X)$ in the preceding paragraph, but Lemma 14 below shows all three definitions to be equivalent in the circumstances which mainly concern us in this paper. We have adopted the definition based on towers because the notion of a tower has a certain intuitive vividness, and to preserve some continuity with [2] and [15]; but the other definitions are sometimes easier to work with (cf. the proofs of Lemmas 20, 21 and 25 below) and one of them might well come to be regarded as the "right" definition at some future time.

We define $D(X)$ to be $\{a \in M: K\langle a\rangle \subseteq X\}$, i.e., intuitively, the set of all men who demand wives from X when an espousal of Γ is sought. (Think of D as standing for "demand.") Let $d(X)$ denote $\| D(X)\|$ and $m_0(X)$ denote $\| X \| - d(X)$. Remembering that M is countable, it is easy to see that (1) is equivalent to the statement that $d(X) \leqslant \| X \|$ for every subset X of W, which is equivalent to saying that

$$m_0(X) \geqslant 0 \qquad \text{for every subset } X \text{ of } W. \tag{2}$$

Thus (2), which is obviously a necessary condition for Γ to have an espousal, is in fact a necessary and sufficient condition if Γ is a society of one of the kinds to which the results of [4, 6–11] are applicable. Our main theorem, Theorem 1, describes a refinement of (2) which is a necessary and sufficient condition for the existence of an espousal in *any* male-countable society.

We can think of $m_0(X)$ as being in some sense the "margin" available in X, i.e., the maximum number of women in X who might conceivably be left unmarried after $d(X)$ wives have been found (if possible) for the men in $D(X)$. Of course, this interpretation of $m_0(X)$ may be over-optimistic when $| X | = | D(X)| = \aleph_0$ because $m_0(X)$ is then $\infty - \infty = \infty$ but, even if we can find a different wife $e(a) \in K\langle a\rangle$ for each man $a \in D(X)$, there is no guarantee that we can do it so as to have infinitely many women in X left unmarried. To overcome this crudity in the definition of m_0, we now construct a transfinite sequence $m_0, m_1, \ldots, m_\omega, \ldots$ of increasingly refined "margin functions" such that (as Lemma 4 will show) $m_0(X) \geqslant m_1(X) \geqslant m_2(X) \geqslant \cdots$ for every $X \subseteq W$. We can think of

$m_0(X), m_1(X), m_2(X), \ldots$ as successive estimates, obtained by looking more and more closely at the constraints involved in finding wives for the men in $D(X)$, of the maximum number of women in X who could be left unmarried to any of these men (cf. Lemma 1 below).

The function $m_\alpha: 2^W \to \mathscr{Q}$ will now be defined by transfinite induction on α. Since m_0 has already been defined, we may proceed by supposing that $\alpha > 0$ and that $m_\theta: 2^W \to \mathscr{Q}$ has been defined for every $\theta < \alpha$. If α is a limit ordinal, define $m_\alpha(X)$ to be $\inf\{m_\theta(X): \theta < \alpha\}$ for every $X \subseteq W$. Now suppose that $\alpha = \gamma + 1$ for some γ. Let a man $a \in M$ be called *γ-constrained* if $K\langle a\rangle \subseteq S$ for some subset S of W such that $m_\gamma(S) < \infty$, and *γ-free* if this is not the case. Let $C_\gamma(X)$ denote the set of all γ-constrained men in $D(X)$, $F_\gamma(X)$ denote the set of all γ-free men in $D(X)$ and $f_\gamma(X)$ denote $\| F_\gamma(X)\|$; and define $m_\alpha(X)$ to be $\tilde{m}_\gamma(X) - f_\gamma(X)$.

THEOREM 1. *Γ has an espousal iff $m_\Omega(X) \geqslant 0$ for every subset X of W.*

Theorem 1 is a slightly modified version of the principal theorem of [2], the difference being that [2] uses the definition of the margin functions m_α proposed in [15], which differs from the one given above. However, we shall prove in this paper that the two definitions are equivalent in the circumstances relevant to our discussion. Theorem 1 could be proved by establishing this equivalence and then appealing to the theorem of [2], but we shall in fact give here a complete self-contained proof of Theorem 1. In so doing, we shall necessarily cover some of the same ground as [2]. For instance, Lemmas 1, 9, 11(i), 18, 25(i) and 26(i) below are very closely related to Theorem 1 of [2] and Lemmas 3.2, 5.1, 3.1, 3.5 and 3.4 of [2], respectively. On the other hand, the methods of proof used here are necessarily somewhat different because they are based on a different definition of the margin functions.

Some other results concerning margin functions will emerge in the course of our investigation. The principal ones have been designated as Theorems 2, 3 and 4. Theorems 3 and 4 deal with the equivalence of different methods of defining margin functions.

The necessity of the non-negative margin condition in Theorem 1. If $A \subseteq M$, define an *A-espousal* in Γ to be a one-to-one function $e: A \to W$ such that $e \subseteq K$. (Intuitively, we are finding wives for the men in A only.) A one-to-one function e from a subset of M into W such that $e \subseteq K$ will be called a *partial espousal* of Γ.

LEMMA 1. *If e is a $D(X)$-espousal in Γ then*

$$\| X \backslash e[D(X)]\| \leqslant m_\alpha(X). \tag{3}$$

Proof. Since $e[D(X)] \subseteq K[D(X)] \subseteq X$, it follows that

$$\| X \backslash e[D(X)] \| \leqslant \| X \| - \| e[D(X)] \| = \| X \| - \| D(X) \| = m_0(X),$$

which proves (3) if $\alpha = 0$. Now suppose that $\alpha > 0$, and assume the inductive hypothesis that Lemma 1 becomes true if α is replaced by any smaller ordinal. Then by the inductive hypothesis $\| X \backslash e[D(X)] \| \leqslant m_\theta(X)$ for every $\theta < \alpha$ and therefore $\| X \backslash e[D(X)] \| \leqslant \inf\{m_\theta(X) : \theta < \alpha\}$, which establishes (3) if α is a limit ordinal. We may therefore now suppose that $\alpha = \gamma + 1$ for some γ. Let $\mathbf{T}$ be an m_γ-constant tower on X such that $\hat{m}_\gamma(\mathbf{T}) < \infty$. Let P be a finite subset of $X \backslash e[C_\gamma(X)]$. Then there is a positive integer j such that $P \subseteq T_j$. If $a \in D(T_j)$ then a is γ-constrained because $K\langle a \rangle \subseteq T_j$ and $m_\gamma(T_j) = \hat{m}_\gamma(\mathbf{T}) < \infty$, and $a \in D(X)$ because $K\langle a \rangle \subseteq T_j \subseteq X$, and consequently $a \in C_\gamma(X)$. Hence $D(T_j) \subseteq C_\gamma(X)$, and consequently $P \subseteq T_j \backslash e[D(T_j)]$. From this fact and the inductive hypothesis applied to the $D(T_j)$-espousal $e_j = e \mid D(T_j)$ we obtain

$$\| P \| \leqslant \| T_j \backslash e[D(T_j)] \| = \| T_j \backslash e_j[D(T_j)] \| \leqslant m_\gamma(T_j) = \hat{m}_\gamma(\mathbf{T}).$$

Hence $\hat{m}_\gamma(\mathbf{T}) \geqslant \| P \|$ for every finite subset P of $X \backslash e[C_\gamma(X)]$, which implies that

$$\hat{m}_\gamma(\mathbf{T}) \geqslant \| X \backslash e[C_\gamma(X)] \|. \tag{4}$$

We have thus proved that (4) holds for every $\mathbf{T} \in \mathfrak{T}(X, m_\gamma)$ such that $\hat{m}_\gamma(\mathbf{T}) < \infty$; and since (4) is trivially true when $\hat{m}_\gamma(\mathbf{T}) = \infty$ it in fact holds for every $\mathbf{T} \in \mathfrak{T}(X, m_\gamma)$. Therefore $\tilde{m}_\gamma(X) \geqslant \| X \backslash e[C_\gamma(X)] \|$, whence, since $m_\alpha(X) = \tilde{m}_\gamma(X) - f_\gamma(X)$ by definition and $f_\gamma(X) = \| F_\gamma(X) \| = \| e[F_\gamma(X)] \|$, we obtain

$$\begin{aligned} m_\alpha(X) &\geqslant \| X \backslash e[C_\gamma(X)] \| - \| e[F_\gamma(X)] \| \\ &\geqslant \| (X \backslash e[C_\gamma(X)]) \backslash e[F_\gamma(X)] \| = \| X \backslash e[D(X)] \|, \end{aligned}$$

and Lemma 1 is proved by transfinite induction on α.

If Γ has an espousal e, then $e_X = e \mid D(X)$ is a $D(X)$-espousal for every subset X of W and therefore, by Lemma 1, $m_\Omega(X) \geqslant \| X \backslash e_X[D(X)] \| \geqslant 0$ for every subset X of W, which establishes the necessity of the condition for Γ to have an espousal in Theorem 1. Its sufficiency will be proved in Section 7.

2. Simple Preliminary Lemmas

Lemma 2. *Let $a, b, c, d \in \mathcal{Q}$.*

(I) *If $c > -\infty$ and $d > -\infty$ then $(a + b) - (c + d) = (a - c) + (b - d)$.*

(II) *If* $b > -\infty$, $c > -\infty$ *and* $a - b = c$ *then* $a = b + c$.

(III) *If* $a < \infty$, $c > -\infty$ *and* $a \leqslant b + c$ *then* $a - c \leqslant b$.

(IV) *If* $b > -\infty$, $c < \infty$ *and* $a + b \geqslant c + d$ *then* $a - d \geqslant c - b$.

These assertions can be almost mechanically checked by considering one by one the possible cases obtained by placing each of a, b, c and (in (I) and (IV)) d in one of the sets $\{-\infty\}$, $\mathscr{Z}$, $\{\infty\}$, where $\mathscr{Z}$ is the set of integers. This involves considering 36 cases in proving (I) and (IV), and 12 cases in proving (II) and (III); but it is reasonably easy to shorten this argument by combining cases in suitable ways, and we leave the details to the reader.

LEMMA 3. $\tilde{g}(X) \leqslant g(X)$.

Proof. Let **T** denote the tower $X, X, X, \ldots$. Then $\mathbf{T} \in \mathfrak{T}(X, g)$, and therefore $\tilde{g}(X) \leqslant \hat{g}(\mathbf{T}) = g(X)$.

LEMMA 4. *If* $\alpha \leqslant \beta$ *then* $m_\alpha(X) \geqslant m_\beta(X)$.

Proof. It is clearly sufficient to show that (i) $m_\kappa(X) \geqslant m_{\kappa+1}(X)$ for every ordinal κ and (ii) $m_\theta(X) \geqslant m_\lambda(X)$ if λ is a limit ordinal and $\theta < \lambda$. But, by Lemma 3,

$$m_\kappa(X) \geqslant \tilde{m}_\kappa(X) \geqslant \tilde{m}_\kappa(X) - f_\kappa(X) = m_{\kappa+1}(X),$$

which establishes (i); and (ii) follows immediately from the manner in which $m_\lambda(X)$ is defined for limit ordinals λ.

COROLLARY 4a. *If* $\alpha \leqslant \beta$ *then every* α*-constrained element of* M *is* β*-constrained.*

LEMMA 5. $\tilde{m}_\alpha(X) \leqslant m_\alpha(X) \leqslant \| X \|$.

Proof. By Lemmas 3 and 4,

$$\tilde{m}_\alpha(X) \leqslant m_\alpha(X) \leqslant m_0(X) = \| X \| - d(X) \leqslant \| X \|.$$

DEFINITION. X is *g-admissible* if $g(Z) \geqslant 0$ for every subset Z of X.

LEMMA 6. *If* $X \subseteq Y$, $\alpha \leqslant \beta$ *and* Y *is* m_β*-admissible then* X *is* m_α*-admissible.*

Proof. This follows from Lemma 4.

Lemma 6 will be needed fairly frequently: for instance, in proving a lemma (Lemma n, say), we may wish to use an earlier lemma (Lemma m, say), one of whose hypotheses is that a certain set X is m_α-admissible, and to see that this hypothesis of Lemma m is satisfied we may have to notice that the statement of Lemma n, or some step already taken in its proof, tells us that some set Y such that $X \subseteq Y$ is m_β-admissible for some $\beta \geqslant \alpha$. All this will tend to be indicated briefly by a phrase like "by Lemmas 6 and m," which may be still further abbreviated to "by Lemma m" in later parts of the paper, when the reader has become accustomed to realizing that Lemma 6 is often needed in such circumstances.

In fact, most of our applications of Lemma 6 will involve only the special cases of the lemma in which either $\alpha = \beta$ or $X = Y$.

LEMMA 7. *If X is g-admissible then $\tilde{g}(X) = \hat{g}(\mathbf{T})$ for some $\mathbf{T} \in \mathfrak{T}(X, g)$.*

Proof. Let $\mathscr{A} = \{\hat{g}(\mathbf{T}) \colon \mathbf{T} \in \mathfrak{T}(X, g)\}$. If X is g-admissible then $\mathscr{A}$ is a non-empty set of non-negative quasi-integers. This clearly implies that $\inf \mathscr{A} \in \mathscr{A}$; and Lemma 7 is proved.

COROLLARY 7a. *If X is g-admissible then g subdues X at $\tilde{g}(X)$.*

Proof. Select a $\mathbf{T} \in \mathfrak{T}(X, g)$ such that $\hat{g}(\mathbf{T}) = \tilde{g}(X)$. For any finite subset P of X, there will exist a j such that $P \subseteq T_j \subseteq X$ and $g(T_j) = \tilde{g}(X)$.

LEMMA 8. *If α is a limit ordinal and X is m_α-admissible then $m_\alpha(X) = m_\theta(X)$ for some $\theta < \alpha$.*

Proof. Let $\mathscr{A} = \{m_\theta(X) \colon \theta < \alpha\}$. By Lemma 4 and the m_α-admissibility of X, $m_\theta(X) \geqslant m_\alpha(X) \geqslant 0$ for every $\theta < \alpha$, and therefore $\mathscr{A}$ is a non-empty set of non-negative quasi-integers. Therefore $\inf \mathscr{A} \in \mathscr{A}$; and Lemma 8 is proved.

LEMMA 9. *If X is uncountable then $m_\alpha(X) = \tilde{m}_\alpha(X) = \infty$.*

Proof. Assume the inductive hypothesis (which is vacuous if $\alpha = 0$) that $m_\theta(Z) = \tilde{m}_\theta(Z) = \infty$ for every $\theta < \alpha$ and every uncountable subset Z of W. Let X be an uncountable subset of W.

To prove that $m_\alpha(X) = \infty$, consider three cases. (i) If $\alpha = 0$ then $m_\alpha(X) = \| X \| - d(X) = \infty - d(X) = \infty$. (ii) If α is a limit ordinal then, since $m_\theta(X) = \infty$ for every $\theta < \alpha$ by the inductive hypothesis, we have $\infty = \inf\{m_\theta(X) \colon \theta < \alpha\} = m_\alpha(X)$. (iii) If α is a successor ordinal $\gamma + 1$ then $\tilde{m}_\gamma(X) = \infty$ by the inductive hypothesis and consequently $m_\alpha(X) = \tilde{m}_\gamma(X) - f_\gamma(X) = \infty$.

If $\mathbf{T}$ is a tower on X, there must be a j for which T_j is uncountable and consequently $m_\alpha(T_j) = \infty$ by the argument in the preceding paragraph. Hence $\hat{m}_\alpha(\mathbf{T}) = \infty$ for every $\mathbf{T} \in \mathfrak{T}(X, m_\alpha)$, and therefore $\tilde{m}_\alpha(X) = \infty$.

LEMMA 10. *If* $\mathbf{T}$ *is a tower on* X *and there are integers* a, b *such that* $a \leqslant g(T_j) \leqslant b$ *for every positive integer* j, *then there is a positive integer* J *such that* $g(T_j) \geqslant \tilde{g}(X)$ *for every integer* $j \geqslant J$.

Proof. If c is the least element of $\{a, a+1, a+2,..., b\}$ which occurs infinitely often in the sequence $g(T_1), g(T_2),...,$ then (i) some infinite subsequence $\mathbf{U}$ of $\mathbf{T}$ is a g-constant tower with $\hat{g}(\mathbf{U}) = c$ and (ii) there is a positive integer J such that $g(T_j) \geqslant c$ for all $j \geqslant J$. Clearly $\mathbf{U} \in \mathfrak{T}(X, g)$, and therefore $\tilde{g}(X) \leqslant \hat{g}(\mathbf{U}) = c \leqslant g(T_j)$ for $j \geqslant J$.

3. MARGINS OF UNIONS AND INTERSECTIONS

Notation. $F_\alpha(\Gamma)$ will denote the set of all α-free men of Γ and $C_\alpha(\Gamma)$ will denote the set of all α-constrained men of Γ. (Thus $F_\alpha(\Gamma) = F_\alpha(W)$, $C_\alpha(\Gamma) = C_\alpha(W)$.)

LEMMA 11. *If* $X \cup Y$ *is* m_α*-admissible then*

(i) $m_\alpha(X \cap Y) + m_\alpha(X \cup Y) \leqslant m_\alpha(X) + m_\alpha(Y)$,

(ii) $\tilde{m}_\alpha(X \cap Y) + \tilde{m}_\alpha(X \cup Y) \leqslant \tilde{m}_\alpha(X) + \tilde{m}_\alpha(Y)$.

Proof. Assume the inductive hypothesis that Lemma 11 becomes true if α is replaced by any smaller ordinal (which is of course a vacuous assumption if $\alpha = 0$), and assume that $X \cup Y$ is m_α-admissible.

Clearly,

$$D(X) \cap D(Y) = D(X \cap Y), \qquad D(X) \cup D(Y) \subseteq D(X \cup Y) \tag{5}$$

and hence $d(X) + d(Y) \leqslant d(X \cap Y) + d(X \cup Y)$. Moreover

$$\| X \| + \| Y \| = \| X \cap Y \| + \| X \cup Y \|.$$

Hence

$$\begin{aligned}(\| X \cap Y \| + \| X \cup Y \|) &- (d(X \cap Y) + d(X \cup Y)) \\ &\leqslant (\| X \| + \| Y \|) - (d(X) + d(Y)),\end{aligned}$$

which by Lemma 2(I) implies that (i) is true if $\alpha = 0$. If α is a limit ordinal, then by Lemmas 6 and 8, $m_\alpha(X) = m_\rho(X)$ and $m_\alpha(Y) = m_\sigma(Y)$ for some $\rho, \sigma < \alpha$. Let $\tau = \max(\rho, \sigma)$. Then $X \cup Y$ is m_τ-admissible by Lemma 6, and so, by the inductive hypothesis and Lemma 4,

$$\begin{aligned} m_\alpha(X \cap Y) + m_\alpha(X \cup Y) &\leqslant m_\tau(X \cap Y) + m_\tau(X \cup Y) \\ &\leqslant m_\tau(X) + m_\tau(Y) \leqslant m_\rho(X) + m_\sigma(Y) \\ &= m_\alpha(X) + m_\alpha(Y), \end{aligned}$$

which establishes (i) if α is a limit ordinal. Next, suppose that $\alpha = \gamma + 1$ for some γ. Since $F_\gamma(Z) = D(Z) \cap F_\gamma(\Gamma)$ for every subset Z of W, it follows from (5) that $F_\gamma(X) \cap F_\gamma(Y) = F_\gamma(X \cap Y)$, $F_\gamma(X) \cup F_\gamma(Y) \subseteq F_\gamma(X \cup Y)$ and therefore

$$f_\gamma(X \cap Y) + f_\gamma(X \cup Y) \geqslant f_\gamma(X) + f_\gamma(Y).$$

Moreover, by the inductive hypothesis combined with Lemma 6,

$$\tilde{m}_\gamma(X \cap Y) + \tilde{m}_\gamma(X \cup Y) \leqslant \tilde{m}_\gamma(X) + \tilde{m}_\gamma(Y).$$

Therefore

$$\begin{aligned}(\tilde{m}_\gamma(X \cap Y) + \tilde{m}_\gamma(X \cup Y)) - (f_\gamma(X \cap Y) + f_\gamma(X \cup Y))\\ \leqslant (\tilde{m}_\gamma(X) + \tilde{m}_\gamma(Y)) - (f_\gamma(X) + f_\gamma(Y)),\end{aligned}$$

which by Lemma 2(I) implies (i). Thus (i) has now been proved in all possible cases.

To prove (ii), observe first that it is obviously true if $\tilde{m}_\alpha(X)$ or $\tilde{m}_\alpha(Y)$ is ∞ and so we may assume that $\tilde{m}_\alpha(X) < \infty$ and $\tilde{m}_\alpha(Y) < \infty$. By Lemmas 6 and 7, there exist $\mathbf{T} \in \mathfrak{T}(X, m_\alpha)$, $\mathbf{U} \in \mathfrak{T}(Y, m_\alpha)$ such that $\hat{m}_\alpha(\mathbf{T}) = \tilde{m}_\alpha(X)$, $\hat{m}_\alpha(\mathbf{U}) = \tilde{m}_\alpha(Y)$. Clearly $T_1 \cap U_1, T_2 \cap U_2, \ldots$ and $T_1 \cup U_1, T_2 \cup U_2, \ldots$ are towers on $X \cap Y$ and $X \cup Y$, respectively. By the m_α-admissibility of $X \cup Y$,

$$m_\alpha(T_j \cap U_j) \geqslant 0, \qquad m_\alpha(T_j \cup U_j) \geqslant 0 \qquad (j = 1, 2, \ldots);$$

and by (i),

$$\begin{aligned}m_\alpha(T_j \cap U_j) + m_\alpha(T_j \cup U_j) \leqslant m_\alpha(T_j) + m_\alpha(U_j) = \hat{m}_\alpha(\mathbf{T}) + \hat{m}_\alpha(\mathbf{U})\\ = \tilde{m}_\alpha(X) + \tilde{m}_\alpha(Y) < \infty \qquad (j = 1, 2, \ldots).\end{aligned} \tag{6}$$

Therefore, by Lemma 10, there are positive integers J, J' such that

$$m_\alpha(T_j \cap U_j) \geqslant \tilde{m}_\alpha(X \cap Y) \qquad (j \geqslant J), \tag{7}$$
$$m_\alpha(T_j \cup U_j) \geqslant \tilde{m}_\alpha(X \cup Y) \qquad (j \geqslant J'); \tag{8}$$

and (ii) follows from (6), (7) and (8).

COROLLARY 11a. *If $X_1, \ldots, X_n$ are subsets of W and $X_1 \cup \cdots \cup X_n$ is m_α-admissible then*

(i) $m_\alpha(X_1 \cup \cdots \cup X_n) \leqslant m_\alpha(X_1) + \cdots + m_\alpha(X_n)$,

(ii) $\tilde{m}_\alpha(X_1 \cup \cdots \cup X_n) \leqslant \tilde{m}_\alpha(X_1) + \cdots + \tilde{m}_\alpha(X_n)$.

Proof. Write $Z_i = X_1 \cup \cdots \cup X_i$. The m_α-admissibility of Z_n implies that $m_\alpha(Z_{i-1} \cap X_i) \geqslant 0$ and $\tilde{m}_\alpha(Z_{i-1} \cap X_i) \geqslant 0$ for $i = 2, 3,..., n$. These inequalities and Lemmas 6 and 11 imply that

$$m_\alpha(Z_i) \leqslant m_\alpha(Z_i) + m_\alpha(Z_{i-1} \cap X_i) \leqslant m_\alpha(Z_{i-1}) + m_\alpha(X_i),$$
$$\tilde{m}_\alpha(Z_i) \leqslant \tilde{m}_\alpha(Z_i) + \tilde{m}_\alpha(Z_{i-1} \cap X_i) \leqslant \tilde{m}_\alpha(Z_{i-1}) + \tilde{m}_\alpha(X_i)$$

for $i = 2, 3,..., n$, which clearly implies (i) and (ii).

LEMMA 12. *Let $M' = \{a \in M: | K\langle a\rangle| \leqslant \aleph_0\}$. Suppose that $X \cap Y = K[M'] \cap Y = \varnothing$ and $X \cup Y$ is m_α-admissible. Then*

(i) $m_\alpha(X \cup Y) = m_\alpha(X) + \| Y\|$,

(ii) $\tilde{m}_\alpha(X \cup Y) = \tilde{m}_\alpha(X) + \| Y\|$.

Proof. In addition to the hypotheses of Lemma 12, assume the inductive hypothesis (which is vacuous if $\alpha = 0$) that the lemma becomes true if α is replaced by any smaller ordinal. We shall also assume that $X \cup Y$ is countable, because otherwise (i) and (ii) follow from Lemma 9.

This last assumption implies that $D(X \cup Y) \subsetneq M'$, which, since $K[M'] \cap Y = \varnothing$, implies that $K\langle a\rangle \cap Y = \varnothing$ for every $a \in D(X \cup Y)$. Therefore $D(X \cup Y) = D(X)$, and consequently

$$\begin{aligned} m_0(X \cup Y) &= \| X \cup Y\| - d(X \cup Y) \\ &= \| X\| + \| Y\| - d(X) = m_0(X) + \| Y\|, \end{aligned}$$

which proves (i) if $\alpha = 0$. Moreover, by the inductive hypothesis and Lemma 6, $m_\theta(X \cup Y) = m_\theta(X) + \| Y\|$ for every $\theta < \alpha$, and therefore

$$\inf\{m_\theta(X \cup Y): \theta < \alpha\} = \inf\{m_\theta(X): \theta < \alpha\} + \| Y\|,$$

which establishes (i) if α is a limit ordinal. Now suppose that α is a successor ordinal $\gamma + 1$. Since $D(X \cup Y) = D(X)$, it follows that $\| D(X \cup Y) \cap F_\gamma(\Gamma)\| = \| D(X) \cap F_\gamma(\Gamma)\|$, i.e.,

$$f_\gamma(X \cup Y) = f_\gamma(X). \tag{9}$$

By the inductive hypothesis and Lemma 6,

$$\tilde{m}_\gamma(X \cup Y) = \tilde{m}_\gamma(X) + \| Y\|; \tag{10}$$

and (i) follows by subtraction of (9) from (10).

Since Corollary 11a and Lemma 5 yield

$$\tilde{m}_\alpha(X \cup Y) \leqslant \tilde{m}_\alpha(X) + \tilde{m}_\alpha(Y) \leqslant \tilde{m}_\alpha(X) + \| Y\|,$$

we can prove (ii) by showing that $\tilde{m}_\alpha(X \cup Y) \geqslant \tilde{m}_\alpha(X) + \| Y \|$, and since this is evident if $\tilde{m}_\alpha(X \cup Y) = \infty$ we shall assume that $\tilde{m}_\alpha(X \cup Y) < \infty$. By Lemma 7, $\tilde{m}_\alpha(X \cup Y) = \hat{m}_\alpha(\mathbf{T})$ for some $\mathbf{T} \in \mathfrak{T}(X \cup Y, m_\alpha)$. For every positive integer j,

$$\tilde{m}_\alpha(X \cup Y) = \hat{m}_\alpha(\mathbf{T}) = m_\alpha(T_j) = m_\alpha(T_j \cap X) + \| T_j \cap Y \| \qquad (11)$$

by (i) and Lemma 6. By the m_α-admissibility of $X \cup Y$ and (11), $0 \leqslant m_\alpha(T_j \cap X) \leqslant \tilde{m}_\alpha(X \cup Y) < \infty$ for every j and therefore, by Lemma 10 applied to the tower $T_1 \cap X$, $T_2 \cap X$,..., there exists a J such that

$$m_\alpha(T_j \cap X) \geqslant \tilde{m}_\alpha(X) \qquad (j \geqslant J). \qquad (12)$$

By (11) and (12),

$$\tilde{m}_\alpha(X \cup Y) \geqslant \tilde{m}_\alpha(X) + \| T_j \cap Y \| \qquad (j \geqslant J). \qquad (13)$$

Since $\tilde{m}_\alpha(X) \geqslant 0$ by the m_α-admissibility of $X \cup Y$ and $T_1 \cap Y$, $T_2 \cap Y$,... is a tower on Y, (13) clearly implies that $\tilde{m}_\alpha(X \cup Y) \geqslant \tilde{m}_\alpha(X) + \| Y \|$; and (ii) is proved.

4. Alternative Characterisations of $\tilde{m}_\alpha$

Lemma 13. *Suppose that X is m_α-admissible. Then $\tilde{m}_\alpha(X) \leqslant q$ iff m_α subdues X below q.*

Proof. By Corollary 7a, m_α subdues X below q if $\tilde{m}_\alpha(X) \leqslant q$. To prove the converse, assume that m_α subdues X below q. Since $\tilde{m}_\alpha(X) \leqslant q$ automatically if $q = \infty$, we shall assume that $q < \infty$. Let $M' = \{a \in M: | K\langle a \rangle | \leqslant \aleph_0\}$. Since $M' \subseteq M$, which is countable, and since $K\langle a \rangle$ is countable for each $a \in M'$, it follows that $K[M']$ is countable. If P is any finite subset of $X \backslash K[M']$ then, since m_α subdues X below q, there is a set V such that $P \subseteq V \subseteq X$ and $m_\alpha(V) \leqslant q$, which, by Lemmas 6 and 12 and the m_α-admissibility of X, yields

$$q \geqslant m_\alpha(V) = m_\alpha(V \cap K[M']) + \| V \backslash K[M'] \| \geqslant \| V \backslash K[M'] \| \geqslant \| P \|.$$

Hence $\| P \| \leqslant q$ for every finite subset P of $X \backslash K[M']$, and consequently $\| X \backslash K[M'] \| \leqslant q$. Since $K[M']$ is countable and $\| X \backslash K[M'] \| \leqslant q < \infty$, it follows that X is countable. We can therefore select a tower $\mathbf{R}$ on X such that all the sets R_j are finite. Since q belongs to the set of those quasi-integers below which m_α subdues X and this set has (by the m_α-admissibility of X) no negative elements, it follows that the least element q_0

of this set exists and satisfies $0 \leqslant q_0 \leqslant q < \infty$. Since m_α does not subdue X below $q_0 - 1$, there is a finite subset P_0 of X such that $m_\alpha(Z) \geqslant q_0$ for every set Z such that $P_0 \subseteq Z \subseteq X$. Since m_α subdues X below q_0, we can for each positive integer j select a set Z_j such that $P_0 \cup R_j \subseteq Z_j \subseteq X$ and $m_\alpha(Z_j) \leqslant q_0$. Letting $U_j = Z_1 \cup \cdots \cup Z_j$ defines a tower $\mathbf{U}$ on X. For each positive integer j, $P_0 \subseteq U_j \cap Z_{j+1} \subseteq X$ and therefore $m_\alpha(U_j \cap Z_{j+1}) \geqslant q_0$: this inequality and Lemmas 6 and 11 imply that

$$m_\alpha(U_{j+1}) + q_0 \leqslant m_\alpha(U_{j+1}) + m_\alpha(U_j \cap Z_{j+1})$$
$$\leqslant m_\alpha(U_j) + m_\alpha(Z_{j+1}) \leqslant m_\alpha(U_j) + q_0 . \tag{14}$$

Since $0 \leqslant q_0 \leqslant q < \infty$, (14) implies that $m_\alpha(U_{j+1}) \leqslant m_\alpha(U_j)$ for every positive integer j, and hence $m_\alpha(U_j) \leqslant m_\alpha(U_1) = m_\alpha(Z_1) \leqslant q_0$ for every positive integer j. But $m_\alpha(U_j) \geqslant q_0$ for each j since $P_0 \subseteq U_j$. Hence $\mathbf{U} \in \mathfrak{T}(X, m_\alpha)$ and $\hat{m}_\alpha(\mathbf{U}) = q_0$, from which it follows that $\tilde{m}_\alpha(X) \leqslant q_0 \leqslant q$.

LEMMA 14. *If X is m_α-admissible, then $\tilde{m}_\alpha(X)$ is the least element of each of the following sets*:

(i) $\{\hat{m}_\alpha(\mathbf{T})\colon \mathbf{T} \in \mathfrak{T}(X, m_\alpha)\}$,

(ii) *the set of those quasi-integers at which m_α subdues X*,

(iii) *the set of those quasi-integers below which m_α subdues X.*

Proof. Lemma 7 ensures that $\tilde{m}_\alpha(X)$ belongs to the set (i), and the definition of $\tilde{g}(X)$ ensures that no smaller quasi-integer could so do. Corollary 7a ensures that $\tilde{m}_\alpha(X)$ belongs to (ii) and Lemma 13 ensures that no smaller quasi-integer could do so. Lemma 13 implies that $\tilde{m}_\alpha(X)$ is the least element of (iii).

5. FURTHER ANALYSIS OF THE SEQUENCE OF MARGIN FUNCTIONS

LEMMA 15. *There exists an ordinal $\gamma < \Omega$ such that $C_\theta(\Gamma) = C_\Omega(\Gamma)$ for $\gamma \leqslant \theta \leqslant \Omega$.*

Proof. If $a \in C_\Omega(\Gamma)$ then $K\langle a\rangle$ is contained in a subset S of W such that $\infty > m_\Omega(S) = \inf\{m_\phi(S)\colon \phi < \Omega\}$, which implies that $m_\phi(S) < \infty$ for some $\phi < \Omega$ and hence that a is ϕ-constrained for some $\phi < \Omega$. For each $a \in C_\Omega(\Gamma)$, select a $\phi(a) \nless \Omega$ such that a is $\phi(a)$-constrained. Then, since $C_\Omega(\Gamma)$ (being a subset of M) is countable, there is a $\gamma < \Omega$ such that $\phi(a) < \gamma$ for every $a \in C_\Omega(\Gamma)$. If $\gamma \leqslant \theta \leqslant \Omega$, then, for each $a \in C_\Omega(\Gamma)$, we have $a \in C_{\phi(a)}(\Gamma) \subseteq C_\theta(\Gamma) \subseteq C_\Omega(\Gamma)$ by Corollary 4a; and consequently $C_\theta(\Gamma) = C_\Omega(\Gamma)$.

LEMMA 16. *There exists an ordinal* $\delta = \delta(X) < \Omega$ *such that* $m_\theta(X) = m_\Omega(X)$ *for* $\delta \leqslant \theta \leqslant \Omega$.

Proof. Let $\mathscr{A} = \{m_\theta(X): \theta < \Omega\}$. For each $p \in \mathscr{A}$, select a $\psi(p) < \Omega$ for which $m_{\psi(p)}(X) = p$. Then, since $\mathscr{A}$ is countable, there is a $\delta < \Omega$ such that $\psi(p) < \delta$ for every $p \in \mathscr{A}$. If $\delta \leqslant \theta \leqslant \Omega$ then, by Lemma 4, $m_\Omega(X) \leqslant m_\theta(X) \leqslant m_{\psi(p)}(X) = p$ for each $p \in \mathscr{A}$, and therefore $m_\Omega(X) \leqslant m_\theta(X) \leqslant \inf \mathscr{A} = m_\Omega(X)$.

LEMMA 17. $m_\Omega(X) = \tilde{m}_\Omega(X) = m_{\Omega+1}(X)$.

Proof. Let $\mathbf{T} \in \mathfrak{T}(X, m_\Omega)$. By Lemmas 15 and 16 there exist countable ordinals $\gamma, \delta_1, \delta_2, \ldots$ such that

(i) $C_\theta(\Gamma) = C_\Omega(\Gamma)$ for $\gamma \leqslant \theta \leqslant \Omega$,

(ii) $m_\theta(T_j) = m_\Omega(T_j)$ for $\delta_j \leqslant \theta \leqslant \Omega$ $(j = 1, 2, \ldots)$.

Let ϵ be a countable ordinal greater than all of $\gamma, \delta_1, \delta_2, \ldots$. Then, by (ii), $m_\epsilon(T_j) = m_\Omega(T_j) = \hat{m}_\Omega(\mathbf{T})$ for $j = 1, 2, \ldots$, which shows that $\mathbf{T} \in \mathfrak{T}(X, m_\epsilon)$ and $\hat{m}_\epsilon(\mathbf{T}) = \hat{m}_\Omega(\mathbf{T})$. Therefore,

$$\tilde{m}_\epsilon(X) \leqslant \hat{m}_\Omega(\mathbf{T}). \tag{15}$$

By (i),

$$f_\epsilon(X) = \| D(X) \backslash C_\epsilon(\Gamma) \| = \| D(X) \backslash C_\Omega(\Gamma) \| = f_\Omega(X). \tag{16}$$

By (15), (16) and Lemma 4,

$$\hat{m}_\Omega(\mathbf{T}) - f_\Omega(X) \geqslant \tilde{m}_\epsilon(X) - f_\epsilon(X) = m_{\epsilon+1}(X) \geqslant m_\Omega(X). \tag{17}$$

This argument proves the truth of (17) for every $\mathbf{T} \in \mathfrak{T}(X, m_\Omega)$, from which it is easily inferred that

$$\inf\{\hat{m}_\Omega(\mathbf{T}): \mathbf{T} \in \mathfrak{T}(X, m_\Omega)\} - f_\Omega(X) \geqslant m_\Omega(X). \tag{18}$$

By (18) and Lemma 3,

$$m_\Omega(X) \leqslant \tilde{m}_\Omega(X) - f_\Omega(X) \leqslant \tilde{m}_\Omega(X) \leqslant m_\Omega(X). \tag{19}$$

Lemma 17 follows from (19) and the definition of $m_{\Omega+1}(X)$.

LEMMA 18. *If* $\theta \geqslant \Omega$ *then* $m_\theta(X) = m_\Omega(X)$.

Proof. The lemma is obviously true for $\theta = \Omega$. Now suppose that $\theta > \Omega$ and assume the inductive hypothesis that $m_\phi(Y) = m_\Omega(Y)$ for every pair ϕ, Y such that $\Omega \leqslant \phi < \theta$, $Y \subseteq W$. Then, since $m_\phi(X) \geqslant m_\Omega(X)$

for $\phi < \Omega$ by Lemma 4 and $m_\phi(X) = m_\Omega(X)$ for $\Omega \leqslant \phi < \theta$ by the inductive hypothesis, it follows that $m_\Omega(X) = \inf\{m_\phi(X): \phi < \theta\}$, which is equal to $m_\theta(X)$ if θ is a limit ordinal. If, on the other hand, θ is a successor ordinal $\sigma + 1$, then $m_\sigma(Y) = m_\Omega(Y)$ for every $Y \subseteq W$ by the inductive hypothesis and consequently $\tilde{m}_\sigma(X) = \tilde{m}_\Omega(X)$, $f_\sigma(X) = f_\Omega(X)$, and hence, by Lemma 17,

$$m_\Omega(X) = m_{\Omega+1}(X) = \tilde{m}_\Omega(X) - f_\Omega(X) = \tilde{m}_\sigma(X) - f_\sigma(X) = m_\theta(X).$$

The basic idea of the above proof is taken from [2], although some differences in detail arise from the different definition of the margin functions employed here.

LEMMA 19. *If X is finite then $m_\gamma(X) = m_0(X)$ for every ordinal γ.*

Proof. The lemma will clearly follow by transfinite induction on γ if we prove that (i) $m_{\kappa+1}(X) = m_\kappa(X)$ for every ordinal κ and (ii) $m_\lambda(X) = m_0(X)$ if λ is a limit ordinal and $m_\theta(X) = m_0(X)$ for every $\theta < \lambda$. But (ii) follows at once from the definition $m_\lambda(X) = \inf\{m_\theta(X): \theta < \lambda\}$. To prove (i), observe first that, by Lemma 5, $m_\kappa(X) \leqslant \| X \| < \infty$ and therefore every element of $D(X)$ is κ-constrained, i.e., $f_\kappa(X) = 0$. Furthermore, if $\mathbf{T}$ is a tower on X then $T_j = X$ for all sufficiently large j: therefore $\hat{m}_\kappa(\mathbf{T}) = m_\kappa(X)$ for every $\mathbf{T} \in \mathfrak{T}(X, m_\kappa)$ and consequently $\tilde{m}_\kappa(X) = m_\kappa(X)$. Hence $m_{\kappa+1}(X) = \tilde{m}_\kappa(X) - f_\kappa(X) = m_\kappa(X)$.

LEMMA 20. *If $X \subseteq Y$ and Y is $m_{\alpha+1}$-admissible then*

$$\tilde{m}_\alpha(X) \leqslant \tilde{m}_{\alpha+1}(X) + \tilde{m}_\alpha(Y). \tag{20}$$

Proof. We will assume that $\tilde{m}_{\alpha+1}(X) < \infty$ and $\tilde{m}_\alpha(Y) < \infty$, because otherwise (20) is obviously true. Let P be a finite subset of X. Then, by Corollary 7a, there exist sets U, V such that

$$P \subseteq U \subseteq X, \quad m_{\alpha+1}(U) = \tilde{m}_{\alpha+1}(X), \quad P \subseteq V \subseteq Y, \quad m_\alpha(V) = \tilde{m}_\alpha(Y).$$

Each $a \in D(U \cap V)$ is α-constrained since $K\langle a\rangle \subseteq U \cap V \subseteq V$ and $m_\alpha(V) = \tilde{m}_\alpha(Y) < \infty$. Therefore $f_\alpha(U \cap V) = 0$, and consequently $m_{\alpha+1}(U \cap V) = \tilde{m}_\alpha(U \cap V)$. Since Y is $m_{\alpha+1}$-admissible, $m_{\alpha+1}(U \cup V) \geqslant 0$. By these remarks and Lemmas 11 and 4,

$$\begin{aligned}\tilde{m}_\alpha(U \cap V) &\leqslant m_{\alpha+1}(U \cap V) + m_{\alpha+1}(U \cup V) \leqslant m_{\alpha+1}(U) + m_{\alpha+1}(V) \\ &\leqslant m_{\alpha+1}(U) + m_\alpha(V) = \tilde{m}_{\alpha+1}(X) + \tilde{m}_\alpha(Y).\end{aligned}$$

Hence, by Lemma 13, m_α subdues $U \cap V$ below $\tilde{m}_{\alpha+1}(X) + \tilde{m}_\alpha(Y)$. Therefore $P \subseteq Z \subseteq U \cap V$ for some Z such that $m_\alpha(Z) \leqslant \tilde{m}_{\alpha+1}(X) + \tilde{m}_\alpha(Y)$. Since $Z \subseteq U \cap V \subseteq X$, our argument has proved that every finite subset P of X is contained in a subset Z of X such that $m_\alpha(Z) \leqslant \tilde{m}_{\alpha+1}(X) + \tilde{m}_\alpha(Y)$, which by Lemma 13 implies (20).

LEMMA 21. *If $X \subseteq Y$, $m_{\alpha+1}(Y) < \infty$ and Y is $m_{\alpha+1}$-admissible then $m_{\alpha+1}(X) = m_{\alpha+2}(X)$.*

Proof. Let P be a finite subset of X. Then, by Corollary 7a, $P \subseteq Z \subseteq X$ for some set Z such that

$$\tilde{m}_{\alpha+1}(X) = m_{\alpha+1}(Z) = \tilde{m}_\alpha(Z) - f_\alpha(Z). \tag{21}$$

Moreover $\tilde{m}_{\alpha+1}(X) \geqslant 0$ since Y is $m_{\alpha+1}$-admissible and $f_\alpha(Z) = \| F_\alpha(Z) \| \geqslant 0$. Hence (21) and Lemma 2(II) give

$$\tilde{m}_\alpha(Z) = \tilde{m}_{\alpha+1}(X) + f_\alpha(Z) \leqslant \tilde{m}_{\alpha+1}(X) + f_\alpha(X), \tag{22}$$

the inequality in (22) being true because $F_\alpha(Z) \subseteq F_\alpha(X)$. By (22) and Lemma 13, P is contained in a subset V of Z such that $m_\alpha(V) \leqslant \tilde{m}_{\alpha+1}(X) + f_\alpha(X)$. This argument proves that every finite subset P of X is contained in a subset V of X such that $m_\alpha(V) \leqslant \tilde{m}_{\alpha+1}(X) + f_\alpha(X)$. Therefore, by Lemma 13,

$$\tilde{m}_\alpha(X) \leqslant \tilde{m}_{\alpha+1}(X) + f_\alpha(X). \tag{23}$$

If $\tilde{m}_{\alpha+1}(X) = \infty$, then $m_{\alpha+2}(X) = \infty$ by the definition of $m_{\alpha+2}(X)$ and $m_{\alpha+1}(X) = \infty$ by Lemma 3 and so $m_{\alpha+1}(X) = m_{\alpha+2}(X)$ as required. We may therefore assume that $\tilde{m}_{\alpha+1}(X) < \infty$. Since $\tilde{m}_\alpha(Y) - f_\alpha(Y) = m_{\alpha+1}(Y) < \infty$, it follows that $\tilde{m}_\alpha(Y) < \infty$. Hence, by Lemma 20, $\tilde{m}_\alpha(X) < \infty$. Moreover, $f_\alpha(X) = \| F_\alpha(X) \| \geqslant 0$. Therefore (23) and Lemma 2(III) imply that $\tilde{m}_\alpha(X) - f_\alpha(X) \leqslant \tilde{m}_{\alpha+1}(X)$, i.e., $m_{\alpha+1}(X) \leqslant \tilde{m}_{\alpha+1}(X)$, and therefore, by Lemma 3, $\tilde{m}_{\alpha+1}(X) = m_{\alpha+1}(X)$. Moreover, each $a \in D(X)$ is $(\alpha + 1)$-constrained since $K\langle a \rangle \subseteq X \subseteq Y$ and $m_{\alpha+1}(Y) < \infty$: therefore $f_{\alpha+1}(X) = 0$. Hence $m_{\alpha+1}(X) = \tilde{m}_{\alpha+1}(X) - f_{\alpha+1}(X) = m_{\alpha+2}(X)$.

THEOREM 2. *If $X \subseteq Y$, $\alpha \leqslant \beta$, $m_\alpha(Y) < \infty$ and Y is m_α-admissible then $m_\alpha(X) = m_\beta(X)$.*

Proof. Let an ordered quadruple (X, Y, α, β) be called *bad* if $X \subseteq Y \subseteq W$, $\alpha \leqslant \beta$, $m_\alpha(Y) < \infty$, Y is m_α-admissible and $m_\alpha(X) \neq m_\beta(X)$. Suppose that at least one bad ordered quadruple exists. Let (X, Y, α, β) henceforward denote a bad ordered quadruple chosen so as to minimize α and, subject to this prior choice of α, chosen so as to minimize β.

If α is a limit ordinal then by Lemma 8 there exist $\theta, \phi < \alpha$ such that $m_\theta(X) = m_\alpha(X), m_\phi(Y) = m_\alpha(Y)$. Let $\psi = \max(\theta, \phi)$. Then, by Lemma 4, $m_\psi(X) = m_\alpha(X) \neq m_\beta(X)$, $m_\psi(Y) = m_\alpha(Y) < \infty$; and Y, being m_α-admissible, is by Lemma 6 m_ψ-admissible. Hence (X, Y, ψ, β) is bad, contradicting our choice of (X, Y, α, β) so as to minimize α. If $\alpha = 0$, then $\| Y \| - d(Y) = m_0(Y) = m_\alpha(Y) < \infty$ and therefore $\infty > \| Y \| \geqslant \| X \|$, which by Lemma 19 implies that $m_\alpha(X) = m_\beta(X)$, contradicting the badness of (X, Y, α, β). Hence α must be a successor ordinal.

Our choice of (X, Y, α, β) so as to minimize β subject to a prior choice of α implies that

$$\text{if } V \subseteq Y \text{ and } \alpha \leqslant \theta < \beta \text{ then } m_\alpha(V) = m_\theta(V), \tag{24}$$

because otherwise a bad ordered quadruple (V, Y, α, θ) with $\theta < \beta$ would exist. Since $\alpha \leqslant \beta$ and $m_\alpha(X) \neq m_\beta(X)$, it follows that $\alpha < \beta$. Since $\alpha < \beta$ and $m_\theta(X) = m_\alpha(X)$ for $\alpha \leqslant \theta < \beta$ by (24) and $m_\theta(X) \geqslant m_\alpha(X)$ for $\theta < \alpha$ by Lemma 4, it follows that $\inf\{m_\theta(X): \theta < \beta\} = m_\alpha(X) \neq m_\beta(X)$, which would contradict the definition of $m_\beta(X)$ if β was a limit ordinal. Hence β is not a limit ordinal and so, since $\alpha < \beta$, we have $\beta = \sigma + 1$ for some $\sigma \geqslant \alpha$. Since Y is m_α-admissible and $m_\alpha(Y) < \infty$, and since $m_\sigma(V) = m_\alpha(V)$ for every $V \subseteq Y$ by (24), it follows that Y is m_σ-admissible and $m_\sigma(Y) < \infty$. If σ was a successor ordinal, these observations would by Lemma 21 imply that $m_\sigma(X) = m_{\sigma+1}(X) = m_\beta(X)$, which, together with the fact that $m_\alpha(X) = m_\sigma(X)$ by (24), would contradict the badness of (X, Y, α, β). Hence σ is not a successor ordinal, which, since $\sigma \geqslant \alpha$ and α is a successor ordinal, implies that $\sigma > \alpha + 1$.

By (24), $m_\alpha(V) = m_\sigma(V)$ for every subset V of X, which clearly implies that

$$\tilde{m}_\alpha(X) = \tilde{m}_\sigma(X). \tag{25}$$

If $a \in D(X)$ then $K\langle a\rangle \subseteq X \subseteq Y$ and, by (24), $m_\sigma(Y) = m_\alpha(Y) < \infty$, so that a is both α-constrained and σ-constrained. Therefore

$$f_\alpha(X) = f_\sigma(X) = 0. \tag{26}$$

By (25) and (26), $m_{\alpha+1}(X) = m_{\sigma+1}(X) = m_\beta(X)$. But $m_{\alpha+1}(X) = m_\alpha(X)$ by (24) and the fact that $\alpha + 1 < \sigma < \beta$. Hence $m_\alpha(X) = m_\beta(X)$, contradicting the badness of (X, Y, α, β). Thus our supposition that there exists a bad ordered quadruple has led to a contradiction, and so Theorem 2 is proved.

Corollary 2A. *If $X \subseteq Y$ and $m_\alpha(Y) < \infty$ and Y is m_α-admissible then $m_\alpha(X) = \tilde{m}_\alpha(X)$.*

Proof. By Theorem 2 (with $\beta = \alpha + 1$) and Lemma 3,

$$m_{\alpha+1}(X) = m_\alpha(X) \geqslant \tilde{m}_\alpha(X) \geqslant \tilde{m}_\alpha(X) - f_\alpha(X) = m_{\alpha+1}(X).$$

COROLLARY 2B. *If* $m_\alpha(X) < \infty$ *and* X *is* m_α*-admissible then* (i) $m_\alpha(X) = \tilde{m}_\alpha(X)$ *and* (ii) $m_\theta(X) = m_\alpha(X)$ *for every* $\theta \geqslant \alpha$.

Proof. Take $X = Y$ in Theorem 2 and Corollary 2A.

DEFINITION. $\tau(X)$ will denote the smallest ordinal θ such that $m_\theta(X) = m_\Omega(X)$.

The next lemma, which lists some of the main properties of the function τ, summarises much of what we have learned in this section. Parts (ii) and (iii) of the lemma indicate that, if W is m_Ω-admissible, the functions m_Ω and τ together determine all of the margin functions m_α. Lemma 22 seems worth including for the sake of these insights, although it will not be subsequently used in this paper.

LEMMA 22.

(i) $\tau(X) < \Omega$,

(ii) $m_\alpha(X) = m_\Omega(X)$ *if* $\alpha \geqslant \tau(X)$,

(iii) $m_\alpha(X) = \infty$ *if* $\alpha < \tau(X)$ *and* X *is* m_Ω*-admissible*,

(iv) $\tau(X) > 0$ *iff* $|X| = \aleph_0$ *and* $m_\Omega(X) < \infty$,

(v) *if* $X \subseteq Y$ *and* Y *is* m_Ω*-admissible and* $m_\Omega(Y) < \infty$ *then* $\tau(X) \leqslant \tau(Y)$.

Proof. Lemma 16 implies (i). The definition of $\tau(X)$ and Lemma 4 imply that $m_\alpha(X) = m_\Omega(X)$ for $\tau(X) \leqslant \alpha \leqslant \Omega$: this observation and Lemma 18 imply (ii). If $\alpha < \tau(X)$ and X is m_Ω-admissible, then $\alpha < \Omega$ by (i), X is m_α-admissible by Lemma 6, and $m_\alpha(X) \neq m_\Omega(X)$ by the definition of $\tau(X)$: hence, by Corollary 2B(ii), $m_\alpha(X)$ must be ∞, which proves (iii). If $\tau(X) > 0$ then $m_0(X) \neq m_\Omega(X)$: therefore $|X| = \aleph_0$ by Lemmas 9 and 19, and, by Lemma 4, $m_\Omega(X) < m_0(X)$, which implies that $m_\Omega(X) < \infty$. Conversely, if $|X| = \aleph_0$ and $m_\Omega(X) < \infty$ then

$$m_0(X) = \|X\| - d(X) = \infty - d(X) = \infty \neq m_\Omega(X)$$

and hence $\tau(X) > 0$. This proves (iv). To prove (v), assume that $X \subseteq Y$ and Y is m_Ω-admissible and $m_\Omega(Y) < \infty$. Then $m_{\tau(Y)}(Y) = m_\Omega(Y) < \infty$ by the definition of $\tau(Y)$. Furthermore, $\tau(Y) < \Omega$ by (i), and Y is $m_{\tau(Y)}$-admissible by Lemma 6. Hence, by Theorem 2, $m_{\tau(Y)}(X) = m_\Omega(X)$, which implies that $\tau(X) \leqslant \tau(Y)$ by the definition of $\tau(X)$.

6. Removal of finitely Many Men or Women from Γ

LEMMA 23. *If $X \cup Y$ is m_α-admissible and $\tilde{m}_\alpha(X) < \infty$ and $m_\alpha(Y) < \infty$ then $m_\alpha(X \cap Y) < \infty$.*

Proof. Since $X \cup Y$ is m_α-admissible, it is clear that $\tilde{m}_\alpha(X \cup Y) \geqslant 0$ and hence, by Corollary 2A (with $X \cap Y$ replacing X) and Lemmas 11 and 3,

$$m_\alpha(X \cap Y) = \tilde{m}_\alpha(X \cap Y) \leqslant \tilde{m}_\alpha(X \cap Y) + \tilde{m}_\alpha(X \cup Y)$$
$$\leqslant \tilde{m}_\alpha(X) + \tilde{m}_\alpha(Y) \leqslant \tilde{m}_\alpha(X) + m_\alpha(Y) < \infty.$$

COROLLARY 23a. *Suppose that W is m_α-admissible and $\tilde{m}_\alpha(X) < \infty$ and $a \in M$. Then $a \in C_\alpha(X)$ iff $K\langle a \rangle$ is contained in some subset S of X such that $m_\alpha(S) < \infty$.*

Proof. Suppose that $a \in C_\alpha(X)$. Then a is α-constrained and so $K\langle a \rangle \subseteq Y$ for some $Y \subseteq W$ with $m_\alpha(Y) < \infty$. Moreover $a \in C_\alpha(X) \subseteq D(X)$ and therefore $K\langle a \rangle \subseteq X$. Since $K\langle a \rangle \subseteq X \cap Y$ and since $m_\alpha(X \cap Y) < \infty$ by Lemma 23, we have shown that if $a \in C_\alpha(X)$ then $K\langle a \rangle$ is contained in a subset S of X with $m_\alpha(S) < \infty$; and the converse of this statement is obvious.

LEMMA 24. *If $X \subseteq Y$ and Y is m_α-admissible and $m_\alpha(X) < \infty$ then there exists a set Z such that $X \subseteq Z \subseteq Y$ and $m_\alpha(Z) \leqslant \tilde{m}_\alpha(Y)$.*

Proof. Amongst all subsets of Y which contain X, let Z be one for which $m_\alpha(Z)$ is as small as possible. (Such a Z exists since Y is m_α-admissible.) By Lemma 7, there exists a $\mathbf{T} \in \mathfrak{T}(Y, m_\alpha)$ such that $\hat{m}_\alpha(\mathbf{T}) = \tilde{m}_\alpha(Y)$. The definition of Z implies that $m_\alpha(Z) \leqslant m_\alpha(X) < \infty$ and also that, for each positive integer j, $m_\alpha(Z) \leqslant m_\alpha(Z \cup T_j)$, whence

$$m_\alpha(Z \cap T_j) + m_\alpha(Z) \leqslant m_\alpha(Z \cap T_j) + m_\alpha(Z \cup T_j)$$
$$\leqslant m_\alpha(Z) + m_\alpha(T_j) \tag{27}$$

by Lemma 11. Since Y is m_α-admissible and $m_\alpha(Z) < \infty$, (27) implies that

$$0 \leqslant m_\alpha(Z \cap T_j) \leqslant m_\alpha(T_j) = \tilde{m}_\alpha(Y) \qquad (j = 1, 2, \ldots). \tag{28}$$

If $\tilde{m}_\alpha(Y) = \infty$ then obviously $m_\alpha(Z) \leqslant \tilde{m}_\alpha(Y)$, as required. If $\tilde{m}_\alpha(Y) < \infty$, then (28) implies that some infinite subsequence $\mathbf{V}$ of $Z \cap T_1, Z \cap T_2, \ldots$ is m_α-constant with $\hat{m}_\alpha(\mathbf{V}) \leqslant \tilde{m}_\alpha(Y)$, so that $\mathbf{V} \in \mathfrak{T}(Z, m_\alpha)$ and $\tilde{m}_\alpha(Z) \leqslant \hat{m}_\alpha(\mathbf{V}) \leqslant \tilde{m}_\alpha(Y)$. But we have seen that $m_\alpha(Z) < \infty$, and therefore, by Corollary 2B, $m_\alpha(Z) = \tilde{m}_\alpha(Z) \leqslant \tilde{m}_\alpha(Y)$.

DEFINITIONS. If $\mathbf{T}$ is a tower and J is a positive integer, $\mathbf{T}^J$ will denote the tower $T_J, T_{J+1}, T_{J+2}, \ldots$.

If $A \subseteq M \cup W$, $\Gamma - A$ will denote the society $(M \backslash A, W \backslash A, K \cap ((M \backslash A) \times (W \backslash A)))$. If $a, b \in M \cup W$, $\Gamma - a$ will denote $\Gamma - \{a\}$ and $\Gamma - a - b$ will denote $\Gamma - \{a, b\}$.

In accordance with the convention introduced in Section 1, we shall throughout our discussion have a society denoted by Γ under consideration, but sometimes additional societies, such as $\Gamma - a$ where $a \in M \cup W$, may also be discussed. We therefore make the convention that symbols such as D, d, m_α, $\hat{m}_\alpha$, $\tilde{m}_\alpha$, F_α, f_α, C_α, whose meanings depend on the society in which they are interpreted, should always be interpreted in the society denoted by Γ unless some indication is given to the contrary; but when necessary such an indication may be given by attaching the name of a society (Δ, say) in which a symbol is to be interpreted as a superscript, e.g. D^Δ, d^Δ, $m_\alpha{}^\Delta$, $\hat{m}_\alpha{}^\Delta$, $\tilde{m}_\alpha{}^\Delta$, $F_\alpha{}^\Delta$, $f_\alpha{}^\Delta$, $C_\alpha{}^\Delta$. As an illustration of these conventions, statement (i) below means $m_\alpha^{\Gamma-A}(X) = m_\alpha{}^\Gamma(X) + |A \cap D^\Gamma(X)|$.

LEMMA 25. *If W is m_α-admissible and A is a finite subset of M then*

(i) $m_\alpha^{\Gamma-A}(X) = m_\alpha(X) + |A \cap D(X)|$,

(ii) $\tilde{m}_\alpha^{\Gamma-A}(Y) = \tilde{m}_\alpha(Y) + |A \cap C_\alpha(Y)|$.

Proof. In addition to the hypotheses of Lemma 25, assume the inductive hypothesis (which is of course vacuously true if $\alpha = 0$) that the lemma becomes true if α is replaced by any smaller ordinal.

Proof of (i). Since

$$m_0^{\Gamma-A}(X) = \| X \| - \| D^{\Gamma-A}(X) \| = \| X \| - \| D(X) \backslash (A \cap D(X)) \|$$
$$= \| X \| - \| D(X) \| + \| A \cap D(X) \| = m_0(X) + |A \cap D(X)|,$$

(i) is true if $\alpha = 0$. Moreover W, being m_α-admissible, is by Lemma 6 m_θ-admissible for every $\theta < \alpha$, so that, by the inductive hypothesis, $m_\theta^{\Gamma-A}(X) = m_\theta(X) + |A \cap D(X)|$ for every $\theta < \alpha$, and therefore

$$\inf\{m_\theta^{\Gamma-A}(X) : \theta < \alpha\} = \inf\{m_\theta(X) : \theta < \alpha\} + |A \cap D(X)|,$$

which implies (i) if α is a limit ordinal.

Now suppose that α is a successor ordinal $\gamma + 1$. Then, by the inductive hypothesis,

$$m_\gamma^{\Gamma-A}(Z) = m_\gamma(Z) + |A \cap D(Z)| \qquad \text{for every} \quad Z \subseteq W, \tag{29}$$

$$\tilde{m}_\gamma^{\Gamma-A}(X) = \tilde{m}_\gamma(X) + |A \cap C_\gamma(X)|. \tag{30}$$

For every $S \subseteq W$, (29) implies that $m_\gamma^{\Gamma-A}(S) < \infty$ iff $m_\gamma(S) < \infty$. Therefore an element of $M \backslash A$ is γ-constrained in $\Gamma - A$ iff it is γ-constrained in Γ, and hence $F_\gamma^{\Gamma-A}(X) = F_\gamma(X) \backslash A$ and therefore

$$f_\gamma^{\Gamma-A}(X) + |A \cap F_\gamma(X)| = f_\gamma(X). \tag{31}$$

Since $C_\gamma(X)$, $F_\gamma(X)$ are disjoint sets with union $D(X)$,

$$|A \cap D(X)| = |A \cap C_\gamma(X)| + |A \cap F_\gamma(X)|; \tag{32}$$

and (i) follows from (30), (31), (32) and the definitions

$$m_\alpha(X) = \tilde{m}_\gamma(X) - f_\gamma(X), \quad m_\alpha^{\Gamma-A}(X) = \tilde{m}_\gamma^{\Gamma-A}(X) - f_\gamma^{\Gamma-A}(X).$$

Proof of (ii). By Lemma 7, there is a $\mathbf{T} \in \mathfrak{T}(Y, m_\alpha)$ such that $\hat{m}_\alpha(\mathbf{T}) = \tilde{m}_\alpha(Y)$. Since A is finite and $D(T_1) \subseteq D(T_2) \subseteq \cdots$, there exist a subset B of A and a positive integer J such that $A \cap D(T_j) = B$ for every $j \geqslant J$. Therefore, by (i), $m_\alpha^{\Gamma-A}(T_j) = m_\alpha(T_j) + |B| = \tilde{m}_\alpha(Y) + |B|$ for $j \geqslant J$ and so $\mathbf{T}^J \in \mathfrak{T}(Y, m_\alpha^{\Gamma-A})$ and $\hat{m}_\alpha^{\Gamma-A}(\mathbf{T}^J) = \tilde{m}_\alpha(Y) + |B|$, which implies that

$$\tilde{m}^{\Gamma-A}(Y) \leqslant \tilde{m}_\alpha(Y) + |B|. \tag{33}$$

If $\tilde{m}_\alpha(Y) < \infty$ then, since $B \subseteq D(T_J) \subseteq D(Y)$ and $m_\alpha(T_J) = \hat{m}_\alpha(\mathbf{T}) = \tilde{m}_\alpha(Y) < \infty$, it follows that $B \subseteq C_\alpha(Y)$ and consequently $B \subseteq A \cap C_\alpha(Y)$, so that (33) implies that

$$\tilde{m}_\alpha^{\Gamma-A}(Y) \leqslant \tilde{m}_\alpha(Y) + |A \cap C_\alpha(Y)|. \tag{34}$$

Since (34) is obviously true if $\tilde{m}_\alpha(Y) = \infty$, it has now been established in general, and it only remains to be proved that

$$\tilde{m}_\alpha(Y) + |A \cap C_\alpha(Y)| \leqslant \tilde{m}_\alpha^{\Gamma-A}(Y). \tag{35}$$

Since (35) is automatically true if $\tilde{m}_\alpha^{\Gamma-A}(Y) = \infty$, we shall assume that $\tilde{m}_\alpha^{\Gamma-A}(Y) < \infty$. We observe first that, by (i) and the m_α-admissibility of W, every subset of W is $m_\alpha^{\Gamma-A}$-admissible. Let P be a finite subset of Y. Let $A \cap C_\alpha(Y) = \{a_1, \ldots, a_r\}$. Since $a_i \in C_\alpha(Y)$, there is an $S_i \subseteq W$ such that $K\langle a_i \rangle \subseteq S_i$ and $m_\alpha(S_i) < \infty$. Let $S = S_1 \cup \cdots \cup S_r$. By Corollary 11a, $m_\alpha(S) \leqslant m_\alpha(S_1) + \cdots + m_\alpha(S_r) < \infty$, and therefore, by (i), $m_\alpha^{\Gamma-A}(S) < \infty$. Moreover $S \cup Y$ is $m_\alpha^{\Gamma-A}$-admissible, and by assumption $\tilde{m}_\alpha^{\Gamma-A}(Y) < \infty$. Hence $m_\alpha^{\Gamma-A}(S \cap Y) < \infty$ by Lemma 23. Furthermore, $(S \cap Y) \cup P$ is $m_\alpha^{\Gamma-A}$-admissible and, by Lemma 5, $m_\alpha^{\Gamma-A}(P) \leqslant \| P \| < \infty$. Hence $m_\alpha^{\Gamma-A}((S \cap Y) \cup P) < \infty$ by Corollary 11a. Therefore, by Lemma 24 and the $m_\alpha^{\Gamma-A}$-admissibility of Y, there exists a $Z \subseteq Y$ such that $(S \cap Y) \cup P \subseteq Z$ and $m_\alpha^{\Gamma-A}(Z) \leqslant \tilde{m}_\alpha^{\Gamma-A}(Y)$. For $i = 1, \ldots, r$ we have $K\langle a_i \rangle \subseteq S_i \subseteq S$

and moreover $K\langle a_i\rangle \subseteq Y$ since $a_i \in C_\alpha(Y) \subseteq D(Y)$, so that $K\langle a_i\rangle \subseteq S \cap Y \subseteq Z$ and therefore $a_i \in D(Z)$. Hence $A \cap C_\alpha(Y) \subseteq D(Z)$, and therefore $A \cap C_\alpha(Y) \subseteq A \cap D(Z)$, which, together with (i), yields

$$m_\alpha(Z) + |A \cap C_\alpha(Y)| \leqslant m_\alpha^{\Gamma-A}(Z) \leqslant \tilde{m}_\alpha^{\Gamma-A}(Y).$$

We have thus proved that, for every finite subset P of Y, there is a set Z such that $P \subseteq Z \subseteq Y$ and $m_\alpha(Z) \leqslant \tilde{m}_\alpha^{\Gamma-A}(Y) - |A \cap C_\alpha(Y)|$. Hence m_α subdues Y below $\tilde{m}_\alpha^{\Gamma-A}(Y) - |A \cap C_\alpha(Y)|$, and so (35) follows from Lemma 13.

LEMMA 26. *If W is m_α-admissible and P is a finite subset of $W\backslash X$ then*

(i) $m_\alpha(X \cup P) = m_\alpha^{\Gamma-P}(X) + |P|$,

(ii) $\tilde{m}_\alpha(X \cup P) = \tilde{m}_\alpha^{\Gamma-P}(X) + |P|$.

Proof. In addition to the hypotheses of Lemma 26, assume the inductive hypothesis that the lemma becomes true if α is replaced by any smaller ordinal.

Proof of (i). Since $D^{\Gamma-P}(X) = D(X \cup P)$, it follows that

$$\begin{aligned} \|X \cup P\| - d(X \cup P) &= (\|X\| + \|P\|) - d^{\Gamma-P}(X) \\ &= (\|X\| - d^{\Gamma-P}(X)) + |P|, \end{aligned}$$

which proves (i) if $\alpha = 0$. If $\alpha > 0$, then, since $m_\theta(X \cup P) = m_\theta^{\Gamma-P}(X) + |P|$ for every $\theta < \alpha$ by the inductive hypothesis, it follows that

$$\inf\{m_\theta(X \cup P) \colon \theta < \alpha\} = \inf\{m_\theta^{\Gamma-P}(X) \colon \theta < \alpha\} + |P|,$$

which proves (i) if α is a limit ordinal.

Now suppose that α is a successor ordinal $\gamma + 1$. Let $K' = K \cap (M \times (W\backslash P))$. Suppose that $a \in C_\gamma(\Gamma)$. Then $K\langle a\rangle$ is contained in a subset S of W such that $m_\gamma(S) < \infty$. By the inductive hypothesis, Corollary 11a and Lemma 5,

$$m_\gamma^{\Gamma-P}(S\backslash P) + |P| = m_\gamma(S \cup P) \leqslant m_\gamma(S) + m_\gamma(P) \leqslant m_\gamma(S) + \|P\| < \infty,$$

and hence $m_\alpha^{\Gamma-P}(S\backslash P) < \infty$. But $K'\langle a\rangle = K\langle a\rangle\backslash P \subseteq S\backslash P$. Hence $a \in C_\gamma(\Gamma - P)$. Conversely, suppose that $b \in C_\gamma(\Gamma - P)$. Then $K'\langle b\rangle$ is contained in a subset S' of $W\backslash P$ such that $m_\gamma^{\Gamma-P}(S') < \infty$. Since $K\langle b\rangle \subseteq K'\langle b\rangle \cup P \subseteq S' \cup P$ and since, by the inductive hypothesis, $m_\gamma(S' \cup P) = m_\gamma^{\Gamma-P}(S') + |P| < \infty$, it follows that $b \in C_\gamma(\Gamma)$. We have thus proved that $C_\gamma(\Gamma) = C_\gamma(\Gamma - P)$; and since $D(X \cup P) = D^{\Gamma-P}(X)$ it follows that

$$f_\gamma(X \cup P) = \| D(X \cup P) \backslash C_\gamma(\Gamma) \| = \| D^{\Gamma-P}(X) \backslash C_\gamma(\Gamma - P) \| = f_\gamma^{\Gamma-P}(X).$$

Moreover, $\tilde{m}_\gamma(X \cup P) = \tilde{m}_\gamma^{\Gamma-P}(X) + | P |$ by the inductive hypothesis. Hence

$$\tilde{m}_\gamma(X \cup P) - f_\gamma(X \cup P) = \tilde{m}_\gamma^{\Gamma-P}(X) - f_\gamma^{\Gamma-P}(X) + | P |;$$

and (i) is proved.

Proof of (ii). Let

$$\mathscr{A} = \{\hat{m}_\alpha^{\Gamma-P}(\mathbf{T}) \colon \mathbf{T} \in \mathfrak{T}(X, m_\alpha^{\Gamma-P})\},$$
$$\mathscr{B} = \{\hat{m}_\alpha(\mathbf{U}) \colon \mathbf{U} \in \mathfrak{T}(X \cup P, m_\alpha)\}.$$

Suppose that $t \in \mathscr{A}$. Select a $\mathbf{T} \in \mathfrak{T}(X, m_\alpha^{\Gamma-P})$ such that $\hat{m}_\alpha^{\Gamma-P}(\mathbf{T}) = t$. Then (i) implies that $m_\alpha(T_j \cup P) = m_\alpha^{\Gamma-P}(T_j) + | P | = t + | P |$ for every positive integer j, and therefore $T_1 \cup P$, $T_2 \cup P, \ldots$ is a tower $\mathbf{U} \in \mathfrak{T}(X \cup P, m_\alpha)$ such that $\hat{m}_\alpha(\mathbf{U}) = t + | P |$. Therefore $t + | P | \in \mathscr{B}$. Conversely, suppose that $u' \in \mathscr{B}$. Select a $\mathbf{U}' \in \mathfrak{T}(X \cup P, m_\alpha)$ such that $\hat{m}_\alpha(\mathbf{U}') = u'$. Since P is finite, there is a positive integer J such that $P \subseteq U_j'$ for every integer $j \geqslant J$. By (i),

$$m_\alpha^{\Gamma-P}(U_j' \backslash P) + | P | = m_\alpha(U_j') = u' \quad (j \geqslant J)$$

and therefore $U_J' \backslash P$, $U_{J+1}' \backslash P$, $U_{J+2}' \backslash P, \ldots$ is a tower $\mathbf{T}' \in \mathfrak{T}(X, m_\alpha^{\Gamma-P})$ such that $\hat{m}_\alpha^{\Gamma-P}(\mathbf{T}') + | P | = u'$, which implies that $u' = t' + | P |$ for some $t' \in \mathscr{A}$. We have thus proved that $\mathscr{B} = \{t + | P | \colon t \in \mathscr{A}\}$. Therefore $\inf \mathscr{B} = \inf \mathscr{A} + | P |$, i.e., (ii) is true.

7. Existence of Espousals

Lemma 27. *If W is m_Ω-admissible and $m_\Omega(V) > 0$ for every set V such that $X \subseteq V \subsetneq W$, then there is an $x \in X$ such that $W \backslash \{x\}$ is $m_\Omega^{\Gamma-x}$-admissible.*

Proof. Suppose first that $| X | > \aleph_0$. Let $M' = \{a \in M \colon | K\langle a \rangle | \leqslant \aleph_0\}$. Since $M' \subseteq M$, which is countable, and since $K\langle a \rangle$ is countable for each $a \in M'$, it follows that $K[M']$ is countable. Therefore we can select an $x \in X \backslash K[M']$. If $Z \subsetneq W \backslash \{x\}$ then, by Lemmas 26 and 12,

$$m_\Omega^{\Gamma-x}(Z) + |\{x\}| = m_\Omega(Z \cup \{x\}) = m_\Omega(Z) + \|\{x\}\|,$$

and consequently $m_\Omega^{\Gamma-x}(Z) = m_\Omega(Z) \geqslant 0$ since W is m_Ω-admissible. Therefore $W\backslash\{x\}$ is $m_\Omega^{\Gamma-x}$-admissible.

Now suppose that $|X| \leqslant \aleph_0$. By Lemma 5 and (since $X \subseteq X \subseteq W$) by the second hypothesis of Lemma 27, $\|X\| \geqslant m_\Omega(X) > 0$. Therefore there exists an infinite sequence $x_1, x_2, \ldots,$ of (not necessarily distinct) elements of X such that each element of X appears at least once in this sequence. Suppose that, for each positive integer j, there is a set U_j such that $x_j \in U_j \subseteq W$ and $m_\Omega(U_j) = 0$. Let $T_j = U_1 \cup \cdots \cup U_j$ $(j = 1, 2, \ldots)$ and $U = U_1 \cup U_2 \cup \cdots$. Then, by the m_Ω-admissibility of W and Corollary 11a, $0 \leqslant m_\Omega(T_j) \leqslant m_\Omega(U_1) + \cdots + m_\Omega(U_j) = 0$, so that $\mathrm{T} \in \mathfrak{T}(m_\Omega, U)$ and $0 = \hat{m}_\Omega(\mathrm{T}) \geqslant \tilde{m}_\Omega(U) = m_\Omega(U)$ by Lemma 17, contradicting the hypothesis that $m_\Omega(V) > 0$ for every set V such that $X \subseteq V \subseteq W$. This contradiction shows that there is an $x = x_j$ which belongs to no set R such that $m_\Omega(R) = 0$. This property of x and the m_Ω-admissibility of W imply that, if $Z \subseteq W\backslash\{x\}$, then $m_\Omega(Z \cup \{x\}) \geqslant 1$ and therefore $m_\Omega^{\Gamma-x}(Z) \geqslant 0$ by Lemma 26, which proves that $W\backslash\{x\}$ is $m_\Omega^{\Gamma-x}$-admissible.

LEMMA 28. *If W is m_Ω-admissible and $a \in M$ then there is an $x \in K\langle a\rangle$ such that $W\backslash\{x\}$ is $m_\Omega^{\Gamma-a-x}$-admissible.*

Proof. Since W is m_Ω-admissible, it follows by Lemma 25 that it is $m_\Omega^{\Gamma-a}$-admissible. If $K\langle a\rangle \subseteq V \subseteq W$ then $a \in D(V)$ and therefore $m_\Omega^{\Gamma-a}(V) = m_\Omega(V) + 1 > 0$ by Lemma 25 and the m_Ω-admissibility of W. Therefore, by Lemma 27, there is an $x \in K\langle a\rangle$ such that $W\backslash\{x\}$ is $m_\Omega^{\Gamma-a-x}$-admissible.

Proof of Theorem 1. In the last paragraph of Section 1, we showed that W is m_Ω-admissible if Γ has an espousal. To prove the converse, assume that W is m_Ω-admissible.

Suppose first that $|M| = \aleph_0$, and let $a_1, a_2, \ldots$ be an enumeration of M. By Lemma 28, there is an $x_1 \in K\langle a_1\rangle$ such that $W\backslash\{x_1\}$ is $m_\Omega^{\Gamma(1)}$-admissible, where $\Gamma(1) = \Gamma - a_1 - x_1 = (M_1, W_1, K_1)$, say. By Lemma 28, there is an $x_2 \in K_1\langle a_2\rangle$ such that $W_1\backslash\{x_2\}$ is $m_\Omega^{\Gamma(2)}$-admissible, where $\Gamma(2) = \Gamma(1) - a_2 - x_2 = (M_2, W_2, K_2)$, say. By Lemma 28, there is an $x_3 \in K_2\langle a_3\rangle$ such that $W_2\backslash\{x_3\}$ is $m_\Omega^{\Gamma(3)}$-admissible, where $\Gamma(3) = \Gamma(2) - a_3 - x_3 = (M_3, W_3, K_3)$, say. This argument may be continued so as to select an infinite sequence $x_1, x_2, \ldots$ of elements of W, and then $\{\langle a_1, x_1\rangle, \langle a_2, x_2\rangle, \ldots\}$ is an espousal of Γ.

If M is finite, let $a_1, \ldots, a_n$ be an enumeration of M, carry out the preceding argument until x_n has been selected, and then $\{\langle a_1, x_1\rangle, \ldots, \langle a_n, x_n\rangle\}$ is an espousal of Γ. Alternatively, when M is finite we can by Lemma 4 argue that, for every subset A of M,

$$\begin{aligned} \|K[A]\| - \|A\| &\geqslant \|K[A]\| - \|D(K[A])\| \\ &= m_0(K[A]) \geqslant m_\Omega(K[A]) \geqslant 0, \end{aligned}$$

so that (1) holds, and therefore Γ has an espousal by the König–Hall theorem.

8. Alternative Definitions of the Margin Functions

Definitions. If $\mathbf{T}$ is a tower on X then $D(\mathbf{T})$ will denote $D(T_1) \cup D(T_2) \cup \cdots$, $F(\mathbf{T})$ will denote $D(X)\backslash D(\mathbf{T})$ and $f(\mathbf{T})$ will denote $\| F(\mathbf{T})\|$. (The reader is warned that these definitions interchange, in the interest of compatibility with other conventions in this paper, the meanings assigned to $D(\mathbf{T})$ and $F(\mathbf{T})$ in [2] and [15].)

Lemma 29. *If W is m_α-admissible and $\mathbf{T} \in \mathfrak{T}(X, m_\alpha)$ then*

$$\tilde{m}_\alpha(X) + \| F(\mathbf{T}) \cap C_\alpha(X)\| \leqslant \hat{m}_\alpha(\mathbf{T}). \tag{36}$$

Proof. Let A be a finite subset of $F(\mathbf{T}) \cap C_\alpha(X)$. Then, for every j, $A \cap D(T_j) = \varnothing$ and therefore, by Lemma 25, $m_\alpha^{\Gamma-A}(T_j) = m_\alpha(T_j) = \hat{m}_\alpha(\mathbf{T})$, which implies that $\mathbf{T} \in \mathfrak{T}(X, m_\alpha^{\Gamma-A})$ and $\hat{m}_\alpha^{\Gamma-A}(\mathbf{T}) = \hat{m}_\alpha(\mathbf{T})$. By this observation and Lemma 25,

$$\hat{m}_\alpha(\mathbf{T}) \geqslant \tilde{m}_\alpha^{\Gamma-A}(X) = \tilde{m}_\alpha(X) + | A |. \tag{37}$$

Now (36) follows from the truth of (37) for every finite subset A of $F(\mathbf{T}) \cap C_\alpha(X)$ and the fact that $\tilde{m}_\alpha(X) \geqslant 0$ since W is m_α-admissible.

Definitions. In Section 1, $m_{\gamma+1}(X)$ was defined to be $\tilde{m}_\gamma(X) - f_\gamma(X)$, where $f_\gamma(X)$ is the number of elements a of $D(X)$ such that no subset S of W with $m_\gamma(S) < \infty$ contains $K\langle a\rangle$. Instead of considering all subsets S of W with the relevant properties, it might arguably have seemed more natural to restrict attention to subsets S of X, thus making our definition of $m_{\gamma+1}(X)$ "depend only on what happens inside X" (or, to be more precise, depend only on the society $(D(X), X, K \mid D(X))$). This idea suggests the following procedure for defining, by transfinite induction on α, a new "αth margin function" l_α: $2^W \to \mathscr{Q}$. First, $l_0(X)$ is defined to be $\| X \| - d(X)$, i.e., l_0 is the same as m_0. If α is a limit ordinal and l_θ has been defined for every $\theta < \alpha$, define $l_\alpha(X)$ to be $\inf\{l_\theta(X): \theta < \alpha\}$. Finally, suppose that α is a successor ordinal $\gamma + 1$ and l_γ has already been defined. Let a man a be called *(X, γ)-free* if there is no subset S of X such that $K\langle a\rangle \subseteq S$ and $l_\gamma(S) < \infty$. Let $F_\gamma^*(X)$ denote the set of all (X, γ)-free men in $D(X)$. Let $f_\gamma^*(X) = \| F_\gamma^*(X)\|$, and define $l_\alpha(X)$ to be $\tilde{l}_\gamma(X) - f_\gamma^*(X)$.

Yet another definition of margin functions is that of [2] and [15], which we now re-state. We again use transfinite induction, beginning by defining a function $p_0 \colon 2^W \to \mathscr{Q}$ by the rule that $p_0(X) = \| X \| - d(X)$ $[= m_0(X)]$. If $\alpha > 0$ and a function $p_\theta \colon 2^W \to \mathscr{Q}$ has been defined for every $\theta < \alpha$, we define $p_\alpha \colon 2^W \to \mathscr{Q}$ by the rule that $p_\alpha(X)$ is $\inf\{p_\theta(X) \colon \theta < \alpha\}$ if α is a limit ordinal, and is $\inf\{\hat{p}_\gamma(\mathbf{T}) - f(\mathbf{T}) \colon \mathbf{T} \in \mathfrak{T}(X, p_\gamma)\}$ if α is a successor ordinal $\gamma + 1$. (The functions p_α thus defined were denoted by m_α in [2] and [15].)

THEOREM 3. *If W is m_θ-admissible for every $\theta < \alpha$ then*

$$m_\alpha(X) = l_\alpha(X) = p_\alpha(X). \tag{38}$$

Proof. Theorem 3 is true for $\alpha = 0$ by the definitions of $m_0(X)$, $l_0(X)$, $p_0(X)$. We shall therefore now assume that $\alpha > 0$, and assume the inductive hypothesis that Theorem 3 becomes true if α is replaced by any smaller ordinal. Assume also the hypothesis of Theorem 3, i.e., that W is m_θ-admissible for every $\theta < \alpha$. This last assumption and the inductive hypothesis imply that $m_\theta(X) = l_\theta(X) = p_\theta(X)$ for every $\theta < \alpha$; and therefore,

$$\inf\{m_\theta(X) \colon \theta < \alpha\} = \inf\{l_\theta(X) \colon \theta < \alpha\} = \inf\{p_\theta(X) \colon \theta < \alpha\},$$

which proves (38) if α is a limit ordinal. We may therefore henceforward suppose that α is a successor ordinal $\gamma + 1$. Then the m_θ-admissibility of W for every $\theta < \alpha$ and the inductive hypothesis imply that

$$m_\gamma(Z) = l_\gamma(Z) = p_\gamma(Z) \qquad \text{for every} \quad Z \subseteq W. \tag{39}$$

Suppose first that $\tilde{m}_\gamma(X) = \infty$. Then $m_\alpha(X) = \tilde{m}_\gamma(X) - f_\gamma(X) = \infty$; and, since (39) implies that $\tilde{l}_\gamma(X) = \tilde{m}_\gamma(X) = \infty$, it follows that $l_\alpha(X) = \tilde{l}_\gamma(X) - f_\gamma{}^*(X) = \infty$. Moreover, if $\mathbf{T} \in \mathfrak{T}(X, p_\gamma)$ then, by (39), $\mathbf{T} \in \mathfrak{T}(X, m_\gamma)$ and $\hat{p}_\gamma(\mathbf{T}) = \hat{m}_\gamma(\mathbf{T}) \geqslant \tilde{m}_\gamma(X) = \infty$. Therefore $\hat{p}_\gamma(\mathbf{T}) - f_\gamma(\mathbf{T}) = \infty$ for every $\mathbf{T} \in \mathfrak{T}(X, p_\gamma)$, and consequently, $p_\alpha(X) = \infty$. Hence (38) is true when $\tilde{m}_\gamma(X) = \infty$.

Now suppose that $\tilde{m}_\gamma(X) < \infty$. Since $\gamma < \alpha$, W is m_γ-admissible. Therefore, by Corollary 23a and (39), a man a belongs to $C_\gamma(X)$ iff $K\langle a \rangle$ is contained in some subset S of X such that $l_\gamma(S) < \infty$. Hence $F_\gamma{}^*(X) = D(X) \backslash C_\gamma(X) = F_\gamma(X)$. Moreover, (39) implies that $\tilde{m}_\gamma(X) = \tilde{l}_\gamma(X)$. Hence,

$$m_\alpha(X) = \tilde{m}_\gamma(X) - f_\gamma(X) = \tilde{l}_\gamma(X) - f_\gamma{}^*(X) = l_\alpha(X). \tag{40}$$

Suppose that $\mathbf{T} \in \mathfrak{T}(X, m_\gamma)$. By Lemma 29,

$$\hat{m}_\gamma(\mathbf{T}) \geqslant \tilde{m}_\gamma(X) + \| F(\mathbf{T}) \cap C_\gamma(X) \|,$$

and, since $F(\mathbf{T}) \subseteq D(X)$, we have

$$f_\gamma(X) \geqslant \| F(\mathbf{T}) \cap F_\gamma(X) \| = \| F(\mathbf{T}) \backslash C_\gamma(X) \|.$$

Hence $\hat{m}_\gamma(\mathbf{T}) + f_\gamma(X) \geqslant \tilde{m}_\gamma(X) + f(\mathbf{T})$. From this inequality, the fact that $f_\gamma(X) = \| F_\gamma(X) \| \geqslant 0$ and our assumption that $\tilde{m}_\gamma(X) < \infty$, it follows by Lemma 2(IV) that

$$\hat{m}_\gamma(\mathbf{T}) - f(\mathbf{T}) \geqslant \tilde{m}_\gamma(X) - f_\gamma(X) = m_\alpha(X). \tag{41}$$

Moreover, by Lemma 7, there is a $\mathbf{U} \in \mathfrak{T}(X, m_\gamma)$ such that $\hat{m}_\gamma(\mathbf{U}) = \tilde{m}_\gamma(X)$. Since $m_\gamma(U_j) = \tilde{m}_\gamma(X) < \infty$, it follows that $D(U_j) \subseteq C_\gamma(X)$ for every j and therefore $D(\mathbf{U}) \subseteq C_\gamma(X)$, and therefore $F_\gamma(X) \subseteq F(\mathbf{U})$, and therefore $f_\gamma(X) \leqslant f(\mathbf{U})$. Hence $\hat{m}_\gamma(\mathbf{U}) - f(\mathbf{U}) \leqslant \tilde{m}_\gamma(X) - f_\gamma(X) = m_\alpha(X)$. From this and the truth of (41) for every $\mathbf{T} \in \mathfrak{T}(X, m_\gamma)$, it follows that

$$m_\alpha(X) = \inf\{\hat{m}_\gamma(\mathbf{T}) - f(\mathbf{T}) \colon \mathbf{T} \in \mathfrak{T}(X, m_\gamma)\},$$

which is by (39) equal to $\inf\{\hat{p}_\gamma(\mathbf{T}) - f(\mathbf{T}) \colon \mathbf{T} \in \mathfrak{T}(X, p_\gamma)\} = p_\alpha(X)$. This conclusion and (40) establish (38).

While the conclusion of Theorem 3 is symmetrical in m_α, l_α, p_α, its hypothesis seems to favour m_θ over l_θ and p_θ. This apparent asymmetry will be removed by Theorem 4 below, for whose proof we shall require

LEMMA 30. *If $\alpha \leqslant \beta$ then $l_\alpha(X) \geqslant l_\beta(X)$ and $p_\alpha(X) \geqslant p_\beta(X)$.*

Proof. The two inequalities asserted are each proved by an adaptation of the proof of Lemma 4, which is almost mechanical except that we prove that $p_\kappa(X) \geqslant p_{\kappa+1}(X)$ by observing that, if $\mathbf{T}$ is the tower $X, X, \ldots$, then $\mathbf{T} \in \mathfrak{T}(X, p_\kappa)$ and consequently

$$p_{\kappa+1}(X) \leqslant \hat{p}_\kappa(\mathbf{T}) - f(\mathbf{T}) \leqslant \hat{p}_\kappa(\mathbf{T}) = p_\kappa(X).$$

THEOREM 4. *The statements "W is m_α-admissible," "W is l_α-admissible" and "W is p_α-admissible" are equivalent.*

Proof. Let $\mathscr{C}_m$ be the class of all ordinals θ for which W is m_θ-admissible, and let $\mathscr{C}_l$, $\mathscr{C}_p$ be analogously defined. We have to prove that $\mathscr{C}_m = \mathscr{C}_l = \mathscr{C}_p$. This is certainly true if each of $\mathscr{C}_m$, $\mathscr{C}_l$, $\mathscr{C}_p$ is the class of all ordinals. If not, let λ be the least ordinal which does not belong to $\mathscr{C}_m \cap \mathscr{C}_l \cap \mathscr{C}_p$. Then W is m_θ-admissible for every $\theta < \lambda$ and consequently, by Theorem 3, $m_\lambda(Z) = l_\lambda(Z) = p_\lambda(Z)$ for every $Z \subseteq W$. From this and the fact that $\lambda \notin \mathscr{C}_m \cap \mathscr{C}_l \cap \mathscr{C}_p$, it follows that $m_\lambda(Z_0) = l_\lambda(Z_0) = p_\lambda(Z_0) < 0$ for some $Z_0 \subseteq W$. Therefore, by Lemmas 4 and 30, $m_\theta(Z_0) < 0$ and $l_\theta(Z_0) < 0$ and $p_\theta(Z_0) < 0$ for every $\theta \geqslant \lambda$, and so each of $\mathscr{C}_m$, $\mathscr{C}_l$, $\mathscr{C}_p$ is precisely the set of all ordinals less than λ.

9. Questions for Future Consideration

It may be fairly easy to extend much of the foregoing theory to male-countable societies (M, W, K) in which $m_\Omega(X)$ is negative for some subsets X of W, and thus obtain *inter alia* some information about *partial* espousals in societies which do not possess espousals. For instance, I would expect it to be easy to prove that, if M is countable, n is a non-negative integer and $m_\Omega(X) \geqslant -n$ for every subset X of W, then there is an A-espousal in (M, W, K) for some subset A of M such that $| M \backslash A | = n$.

An arguably better characterization of male-countable societies with espousals than Theorem 1 might be a demonstration that every male-countable society has either an espousal or some particular kind of substructure which fairly obviously precludes an espousal. The following is a suggestion on these lines. Adopting a common convention, we assume ordinal numbers to be defined so that each ordinal is the set of all smaller ordinals. Define a *female sequence* in a society (M, W, K) to be a transfinite sequence of distinct women, i.e., a one-to-one function from an ordinal number into W. We shall define a quasi-integer $\mu(f)$, which might be called the *margin* of a female sequence f and which, roughly speaking, measures the largest number of women we could hope to leave unmarried in $\operatorname{rge} f$ after working along the sequence f term by term trying at each stage to ensure that wives have been found for all men who demand them from amongst the set of women so far considered. Our definition proceeds by transfinite induction on $\operatorname{dom} f$. Define $\mu(f)$ to be 0 if $\operatorname{dom} f = 0$, i.e., if $f = \varnothing$. If $\operatorname{dom} f > 0$ and $\mu(f')$ has been defined for female sequences f' with $\operatorname{dom} f' < \operatorname{dom} f$, define $\mu(f)$ to be

(i) $\mu(f_\gamma) + 1 - \| D(\operatorname{rge} f) \backslash D(\operatorname{rge} f_\gamma) \|$ if $\operatorname{dom} f$ is a successor ordinal $\gamma + 1$,

(ii) $\liminf_{\theta \to \lambda} \mu(f_\theta) - \| D(\operatorname{rge} f) \backslash \bigcup_{\theta < \lambda} D(\operatorname{rge} f_\theta) \|$ if $\operatorname{dom} f$ is a limit ordinal λ,

where f_γ, f_θ denote $f | \gamma, f | \theta$, respectively, and $\liminf_{\theta \to \lambda} h(\theta)$ means $\sup\{\inf h[\lambda \backslash \theta]: \theta < \lambda\}$ for any function $h: \lambda \to \mathscr{Q}$. Intuitively, (i) expresses the idea that, when the men in $D(\operatorname{rge} f_\gamma)$ have been married to women in $\operatorname{rge} f_\gamma$ with $\mu(f_\gamma)$ such women left unmarried, then adding $f(\gamma)$ to these unmarried women gives us $\mu(f_\gamma) + 1$ women amongst whom to find wives for the men in $D(\operatorname{rge} f) \backslash D(\operatorname{rge} f_\gamma)$, which we might hope to be able to achieve leaving $\mu(f_\gamma) + 1 - \| D(\operatorname{rge} f) \backslash D(\operatorname{rge} f_\gamma) \|$ of these women still unmarried. We leave the reader to interpret (ii). It seems reasonable to ask whether a male-countable society possesses an espousal iff $\mu(f) \geqslant 0$ for every female sequence f in that society. If so, then in each male-countable society (M, W, K) with no espousal, there is a female sequence f with negative margin, and the society $(D(\text{rge} f), \text{rge} f, K \mid D(\text{rge} f))$ might be considered as a part of (M, W, K) whose shape or structure fairly obviously prevents (M, W, K) from having an espousal.

The existence problem for espousals in male-uncountable societies may call for the introduction of some new ideas. The absence, noted in [5, Proposition 5.5] and [2, Sect. 7], of an espousal of $(\Omega\backslash\omega, \Omega, \{(\alpha, \beta): \omega \leqslant \alpha < \Omega, \beta < \alpha\})$ seems difficult to "catch" by any conjectural characterization of societies with espousals on lines *closely* resembling those given and proposed, respectively, in Theorem 1 and the preceding paragraph for male-countable societies. To seek a characterization of male-uncountable societies with espousals, one should probably try to identify as wide as class of male-uncountable societies without espousals as one can, and then try to prove that all others have espousals. In a sense, this type of thinking applied to male-countable societies led to the discovery of Theorem 1.

A further interesting question, discussed in some detail in the latter part of [15], is whether the ideas in this paper may have applications in other areas of infinite combinatorics, or, more generally, infinite mathematics.

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